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THE ELEMENTS

OF

THE THEORY

OF

ASTRONOMY.

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By J.ohn HYMERS, B.D.

FELLOW AND TUTOR OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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38	6	east	west.
73	2 from bottom	$-\frac{1}{2}e^2, -c$	$+\frac{1}{2}e^2, +c$.
78	8 from bottom	$-c$	$+c$.
128	2	36″	56″.
139	4	dimunition	diminution.
143	15	atmosphere	atmospheric.
144	7	SOz	SOZ.
152	1 and 2	$\frac{1}{2}$	$\frac{1}{4}$.
153	9	Δ	$\frac{n}{\Delta}$.
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ASTRONOMY.

CHAPTER I.

INTRODUCTION.

First view of the Heavens, and of the Earth's surface. The Earth's round figure.

1. ASTRONOMY is that branch of Natural Philosophy, which treats of the apparent and real motions of the heavenly bodies. In that division of the science which forms the subject of this Treatise, both of these are investigated from observation and calculation, and in *Physical Astronomy*, the latter are accounted for on mechanical principles.

2. The starry heavens appear, to a spectator on the Earth's surface, to be a vast spherical vault, of which his eye forms the center, and in which the heavenly bodies are situated. It is certain that this appearance is fallacious, and that the stars are not all at the same distance from us; only the eye, being unable to judge of the distance of very remote bodies, and having no scale to compare it with, supposes them all equally distant. When therefore for the purpose of explanation we speak of the celestial vault in which the stars are fixed, we mean an *imaginary* spherical surface of vast dimensions concentric with the eye of the spectator, to which the places of the stars are referred by lines drawn from them to the eye of the spectator.

3. The Earth, at first sight, appears to one of its inhabitants to be a plane, extended indefinitely on all sides, and sustaining the celestial vault; but it is soon perceived that this cannot be the case, and that, as the Sun and the other heavenly bodies pass under it, the Earth must have a limit both laterally and also downwards; and its round figure, besides being suggested by the analogous forms of the Sun and Moon, is immediately inferred from the fol-

lowing obvious considerations. 1st, When a ship approaches land from any quarter, the topmast first becomes visible to a spectator on the shore, and the hull comes in sight last, which, being the largest part, ought to be first seen, if the Earth were a plane. 2nd, The surface of the sea, when viewed from the deck of a ship or the mast-head, is not seen to lose itself in distance and mist, but to be terminated by a clear, sharp, well-defined circle having the spectator in the center, exactly such as would be swept out on the surface of a sphere by a line touching it and passing through a fixed point a little elevated above the surface of the sphere. 3rd, In lunar eclipses, when the Earth is interposed between the Sun and Moon, the Earth's shadow upon the Moon's disk appears round, in all positions of the Earth; and lastly, the Earth has been sailed round in various directions. We conclude therefore that, like the other heavenly bodies, the Earth has the figure of a globe (differing very little from a sphere, as will be seen) every where isolated in space, and surrounded by the heavens.

Fixed Stars; division of them into Constellations.

4. By far the greater part of the heavenly bodies, though collectively in motion, seem never to change their relative situation, that is, their distance from one another; whence they have been called *Fixed Stars.* The angle which the line joining any two of them subtends, is the same in all parts of the Earth; from which it is evident that the distance of any two points on the Earth's surface is evanescent, compared with the distance of the fixed stars from the Earth, and that the Earth is in reality but a point in space. This causes a spectator, wherever he is placed, to find himself in the center of the celestial movements; and therefore, instead of supposing the center of the superficies in which fixed stars appear to lie, to be the eye of the spectator, we may suppose it to be the center of the Earth.

5. The Stars, for the purpose of being conveniently referred to, are divided into groups called Constellations, each distinguished by a name, which is also applied to the portions of the heavens they respectively occupy. The stars in

each group are denoted by the Greek letters α, β, γ, &c. nearly in the order of their apparent magnitude; thus γ Draconis is a star of the third magnitude in the constellation Draco; stars of the fifth magnitude are barely discernible by the naked eye. By magnitude is here meant brilliancy; for the stars, observed with telescopes of the strongest magnifying powers, have no apparent diameters; the different magnitudes which they appear to have when viewed by the naked eye, or through inferior telescopes, being an optical illusion caused by the radiation of light. About two thousand stars only are visible to the naked eye; but it is computed that nearly a hundred millions come within the reach of telescopic vision.

Rational and Sensible Horizon. Dip of the Horizon.

6. If the Plumb-line at any place, that is, a line perpendicular to the plane of standing water, be produced indefinitely both ways, it meets the celestial sphere in two points, of which that above our heads is called the *Zenith*, and the opposite one the *Nadir*. A plane perpendicular to this line at the surface of the Earth, is called the *Sensible Horizon*, and determines the visible portion of the heavens; a plane parallel to the former drawn through the Earth's center, is called the *Rational Horizon;* as the radius of the Earth subtends no perceptible angle at the distance of the stars, the circles in which the celestial vault is cut by these planes, may be supposed to coincide, and any errors introduced by this supposition with respect to bodies at which the Earth's radius subtends a finite angle, may be afterwards taken into account and corrected. Thus in fig. 1, Plate II, let Z be a place on the Earth's surface; then BZ, the direction of the plumb-line at that place, being in the direction of gravity, will pass nearly through the Earth's center T; the plane REH passing through T perpendicular to ZT, produced indefinitely on all sides, is the rational Horizon of the place Z; and the plane AZA' drawn through Z parallel to REH, and consequently touching the Earth's surface at Z, is the sensible horizon of Z. If the eye of the spectator be above the surface of the Earth, as at B, the angle BAZ, in which the line through B touching the surface is inclined to the sensible horizon at Z, is called the *Dip of the Horizon;*

and can be calculated from *BZ*, and the Earth's radius, for the purpose of correcting observed altitudes.

General motion of the heavenly bodies in parallel planes.

7. If the heavenly bodies are attentively observed, they appear to describe circles in planes parallel to each other, about a common axis which meets the heavens in two points called the *North* and *South Poles.* To an inhabitant of the northern hemisphere the stars situated near the north pole, which are called *Circumpolar Stars*, never set but continue visible during the whole of their circular course; others more to the south ascend from the east side of the horizon, and after having attained their greatest height, which is called *Culminating*, descend and disappear in the west, completing only a part of the circle above the horizon. To an inhabitant of the southern hemisphere, the heavens present a spectacle the reverse of this; the stars near the south pole are seen to describe complete circles, whilst those more to the north are only seen in part of their course, or if near the north pole never rise above the horizon.

8. The parallel circular motion of the heavens may be observed by an instrument, such as is represented in fig. 1, Plate I, consisting of a circle *PQA* moveable round an axis *NA*, which coincides with one of its diameters. To the circle is attached a telescope *TQ*, moveable round the center *C* in the plane of the circle. If the axis *NA* be fixed in a proper position, viz. parallel to the common axis about which the heavens revolve, which may be done in a few trials, and the telescope be pointed towards a fixed star *S*, it will be found that the star may be seen through the telescope as long as it continues above the horizon, by making the circle revolve without disturbing the place of the telescope in the plane *PQA*; therefore, as the telescope describes the surface of a right cone round *NA*, the curve in which this surface cuts the celestial sphere whose center is *C*, or the path which the star *S* describes, is a circle. Any other star *s* may be observed in its course, by bringing the telescope into the direction *Cs* without altering the position of *NA*; therefore circles are described by all the stars in planes

perpendicular to *NA*, and having their centers situated in that line*.

9. The angular motion round *NA* is uniform and is the same for all stars; for if the telescope be made to point to a star on one day and be left in that position, it will point to the same star on the next day, after a certain interval, which remains the same at whatever time the first observation is made, and whatever star is chosen.

R Hence the stars describe circles whose planes are parallel to one another and inclined to the horizon, with a uniform angular motion from east to west, all completing their revolutions in the same time; and each star invariably rises and sets in the same points of the horizon.

Equator defined; mode of ascertaining a body's place relative to it.

R 10. The great circle of the celestial sphere, whose plane is parallel to the planes in which the stars move, or perpendicular to the axis passing through the north and south poles, is called the *Equinoctial* or *Equator*.

Declination circles are great circles whose planes pass through the north and south poles; they are perpendicular to the equator, or *Secondaries* to it, and are sometimes called *Meridians*.

R 11. The *Declination* of a star is its angular distance from the equator, measured on the arc of a declination circle intercepted between the star and the equator; it is denominated north and south, according as the star is situated on the north or south side of the equator, and is measured from 0^0 to 90^0. The *Polar Distance* is the angular distance from the pole, or the complement of the declination.

The *Right Ascension* of a star is the arc of the equator, intercepted between a certain fixed point in the equator, called the first point of Aries, and the declination circle passing through the star; right ascension is measured through 360^0

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* This rude method for exhibiting the circular path of a star would succeed very imperfectly on account of the star being unequally raised in different positions by Refraction.

from west to east, that is, in a direction *contrary* to the diurnal motion of the stars.

Thus right ascension and declination are arcs of great circles (or spherical co-ordinates) which determine the position of a point on the celestial sphere, with respect to the first point of Aries and the equator; in the same manner as its two co-ordinates determine the position of a point in a plane, with respect to the origin and one of the axes.

Meridian of a place, and a Star's Hour Angle, defined.

R 12. The *Meridian* of a place on the Earth's surface is the great circle passing through the Pole of the heavens and the zenith of the place; it is perpendicular to the horizon, which it intersects in a line called the *Meridian Line*. The meridian line cuts the horizon in the *north* and *south* points, the former being nearest to the visible pole when the spectator is in north latitude.

13. The angle between the meridian of a place and a declination circle passing through a star, is called the star's *Hour Angle*, because, the stars revolving uniformly round the axis of the heavens, the angle between these circles varies as the time of a star's moving from one to the other.

R The stars appear to revolve round the axis of the heavens, or to describe an angle of 360^{0}, in an invariable interval of $23^{h}.56^{m}$ *mean time*, that is, such as is shewn by a well regulated clock; this is divided into 24 equal parts called *Sidereal Hours*, consequently the stars revolve through 15^{0} in one sidereal hour, and through 1^{0} in four sidereal minutes.

Mode of ascertaining a body's place relative to the Horizon.

R 14. *Vertical Circles* are circles whose planes pass through the zenith and nadir; of these the *Prime Vertical* is perpendicular to the meridian and cuts the horizon in the *east* and *west* point.

Since the equator and prime vertical are each perpendicular to the meridian, their intersection is perpendicular to that plane, and the same is true of the intersection of the prime vertical and horizon; hence these three planes intersect in the same line.

R 15. The *Altitude* and *Zenith Distance* of a star are its respective distances from the horizon and zenith, measured on the arc of a vertical circle passing through it. Hence the altitude together with the zenith distance of a star = 90°.

The *Azimuth* of a star is the angle between a vertical circle passing through it, and the meridian of the place, and is measured by the intercepted arc of the horizon, being denominated north or south according as it commences at the north or south point of the horizon. Thus altitude and azimuth are the spherical co-ordinates which determine the position of a point with respect to the meridian and horizon of any place.

Terrestrial Latitude and Longitude.

16. The section of the Earth's mean surface made by the plane of the meridian, is called the *Terrestrial Meridian*, and that made by the plane of the equator, is called the *Terrestrial Equator*. Also the diameter of the Earth which coincides with the imaginary axis of the heavens is called the Earth's axis, and its extremities the poles of the Earth.

R 17. The *Latitude* of a place on the Earth's surface is the angular distance of its zenith from the equator, measured by the intercepted arc of the meridian; the complement of this, or the distance of the zenith from the pole, is the *Co-Latitude.*

Since the latitude + the zenith distance of the pole = 90° = the altitude of the pole + the zenith distance of the pole; therefore the latitude of any place is equal to the altitude of the pole at that place.

The *Longitude* of a place on the Earth's surface is the angle between the meridian of that place, and another, called a *first* meridian, passing through a given place; it is measured east and west from the first meridian, along the equator, through 180°.

Illustration of the preceding definitions by means of Diagrams.

18. First to illustrate the definitions of the great circles of the sphere, let T be the center of the Earth, fig. 2, situated in the center of the spherical surface $ZPRA$ (of which the black lines represent the eastern hemisphere, and the dotted lines the western, the circle $PZHN$ being in the

plane of the paper) in which the stars appear to be fixed, and which, by its revolution about the axis AP, produces the phenomenon of the diurnal motion. Let z be a place on the Earth's surface, then zT is the direction of the plumb-line at z, and meets the heavens in Z and N, the zenith and nadir of z. A plane drawn through z at right angles to ZN would be the sensible horizon of z; the great circle made by the parallel plane $HWRE$ drawn through T, is the rational horizon of z. P and A are the north and south poles of the heavens, p and a those of the Earth, pa being the Earth's axis, coinciding in direction with the axis of the heavens. $PZHN$ the great circle passing through the pole and zenith, is the meridian of z, its plane cutting the Earth according to the terrestrial meridian zqa, and intersecting the horizon in the meridian line HR, the extremities of which are H the south, and R the north point. The great circle $ZWNE$, drawn through the zenith perpendicular to the meridian, is the prime vertical, and intersects the horizon in E and W, the east and west points. The great circle EQW is the equator, its plane being perpendicular to PA, and intersecting the Earth in the terrestrial equator qet. ZQ, the distance of the zenith from the equator, which is proportional to zq and the angle ZTQ, is the latitude of z, and is equal to PR, the elevation of the pole above the horizon, since each of the arcs ZR, PQ, subtends a right angle; ZP or QH is the co-latitude.

19. Next to illustrate the definitions respecting the places and motions of the Stars. The small circle LDM, whose plane is parallel to the equator, and on the north side thereof, is the circle of daily motion of all stars situated in it, and whose declination is equal to QM; they rise at D, culminate at M, and set at O, being invisible through the smaller portion of their course OLD. Stars situated in the equator rise in the east and set in the west point, and are above the horizon during exactly half their course; those situated south of the equator, as in the parallel GF, are invisible for the greater part of their course; gf is the parallel of a circumpolar star, which is never below the horizon; it crosses the meridian at g and f, which passages

are called respectively its *superior* and *inferior transits*, the interval between them being half the time of a whole revolution. The meridian, as is manifest, intersects the diurnal circle of every star in that point whose height above the horizon is greatest, and divides the arc of the horizon, contained between the points where any star rises and sets, into two equal parts.

20. It may be here useful to compare some of the phenomena presented by different stars at the same place. First it appears that, relative to a place situated in north latitude, a star having north declination is above the horizon for the greater part of its course, and that this portion increases with the declination; when the declination is equal to the co-latitude of the place, the star's diurnal circle at its lowest point, just touches the horizon; when greater, as Uf, which exceeds UR, the diurnal circle lies entirely above the horizon. For stars having south declination, the larger part of the diurnal circle is below the horizon; and if the declination QG exceed the co-latitude QH, the diurnal path is wholly invisible.

As the angular motion round the axis of the heavens is uniform and the same for all stars, the velocity of any star in its diurnal circle is proportional to the radius of that circle, i. e. to the cosine of the star's declination; hence stars in the equator move quickest.

The place of the north pole P is not marked by any star; if it were, that star would be absolutely stationary; the star α Ursæ Minoris, called also Polaris or the pole star, is distant only about $1^{\circ}.33'$ from the pole, and consequently describes a very small circle with a slow motion.

21. Next to compare some of the phenomena presented by the same body at different places. The appearances described above are relative to a particular place z; in order to compare the phenomena produced by the same body at different places, suppose a spectator, keeping on the same meridian, to pass from z to a place whose zenith is Z', and meridian line $H'R'$ at right angles to $Z'T$; the star whose diurnal circle is fg now makes its superior transit south

of him, instead of making both transits north; the diurnal circle *ML* is wholly visible, instead of being partly below the horizon; whilst *GF* has quite disappeared, having before been partly seen. Also the elevation *PR′* of the pole, and consequently of the pole star, above the horizon, has increased, an appearance always observed by persons proceeding northwards, and which could have no existence if the Earth's surface were plane; for in that case, all places would have the same horizon, and lines drawn from any point in it to the pole, on account of the immense distance of the fixed stars, would be parallel; i. e. the elevation of the pole would be invariable. A person may always obtain a rough determination of his latitude by taking the altitude of the Pole star.

22. If a spectator at *z* measure the angles between the plumb-line, and the lines joining his eye and a star at *g* and *f* when crossing his meridian, their difference will give the angle which *gf* subtends at *z*; if now the angle which *gf* subtends at any other point of the same meridian be similarly observed, it will be found to be identical with its former value; this shews that the variation in the position of the vertex of the triangle whose base is *fg*, is too small to produce any sensible effect on the vertical angle; and it is one of the many proofs that may be alleged that the dimensions of the Earth are evanescent compared with the distance of the fixed stars.

23. Again, let the point ♈, fig. 3, be the first point of Aries, or that fixed point in the equator from which right ascension is measured; through a star *S*, draw the declination circle *PSD*, and the vertical circle *ZSN*; then *DS* is the north declination of the star, ♈*D* its right ascension, *PS* its north-polar distance, $\angle ZPS$ its hour angle, *NS* its altitude, *ZS* its zenith distance, and *HN* or $\angle HZN$ its south azimuth.

Reckoned time different at different places.

24. Also let the planes of the declination circles *PQA*, *PDA*, intersect the Earth's surface in the meridians *pqa*, *pma*, and suppose *pqa* to be that meridian from which

longitude is measured, which in England is the meridian of Greenwich; then *qm* or $\angle qpm$ is the east longitude of all places on the meridian *pm a*, and *mn* is the north latitude of all places on the parallel *nr*; and, the heavens revolving about *AP* from east to west, that is, in the direction *E* ♈ *Q*, while the Earth remains fixed, any heavenly body will cross the plane *PDA*, which is the plane of the meridian of *m*, before it crosses *PQA*, which is the corresponding plane for *q*; and the passage will happen one hour earlier for every 15° in the angle of inclination of the planes, that is, one hour earlier for every 15° of longitude, that *m* is east of *q*. Suppose, for instance, *qm* to be 30°, then when the Sun is on the meridian of *q*, and it is consequently noon there, it will be two o'clock in the afternoon at *m*; and at places whose longitude is 30° west of *q*, it will be ten o'clock in the morning.

25. Hence a watch, which shews 12^h when the Sun is on the meridian of *q*, will, for every 15° of longitude that it is carried east or west of *q*, be one hour too slow or one hour too fast; and by means of it a person may determine his longitude, viz. by observing the time shewn by it when the Sun crosses his new meridian. If when the Sun was on his meridian, a person could start eastwards or westwards and traverse the Earth's equator at the rate of 5° an hour, it is clear the Sun would make its subsequent transits over his meridian after intervals of 18^h or 36^h respectively; so that in going round the Earth, which would be effected in 72^h, he would reckon 4 or 2 days, whilst a person stationary would only reckon 3. This is the reason why a ship, in sailing round the world, counts one day too much or too little according as she starts eastwards or westwards.

Clock, Transit, and Circle. Catalogue of Stars formed.

26. Altitudes, zenith distances, and the angular distance between two bodies or two parts of the same body can be observed with proper instruments. In an Observatory the best graduated instruments are fixed in the plane of the meridian, and the altitudes observed with them are meridian altitudes, which are those used in the most important calculations. Azimuths are not often resorted to, on account of

Observatories not being usually furnished with large fixed Instruments proper for observing them. We shall, in a succeeding Chapter, describe various astronomical instruments; here it is only necessary to allude to the most important ones, which are, the *sidereal clock*, the *transit instrument*, and the *mural circle;* the use of the first two is to determine the exact time of transit of the heavenly bodies over the meridian, that of the latter to measure their zenith distances at the instant of transit.

27. We may consider, as has been stated, that the period in which the diurnal revolution of the heavens is performed, never alters; in the interval between a star's leaving the meridian of any place and returning to it, the nicest observations can detect no variation; this (which is different from the solar day, or interval between two successive noons, a variable duration as will be seen) is called a *sidereal day*, and serves as a measure of time, forming an important element in the system of the universe; it is divided into equal portions, called hours, minutes, and seconds, like the solar day; beginning at any place, when the first point of Aries is on the meridian of that place.

A Clock so adjusted that its index completes a circuit in a sidereal day, pointing to $0^h. 0^m. 0^s$ when the first point of Aries is on the meridian, and having the circle on its dial-plate divided into 24 equal spaces, and which beats sidereal seconds, is called a sidereal clock. Hence when the Transit instrument is truly adjusted, and the clock goes accurately, at the instant the first point of Aries crosses the meridian, the index of the clock points $0^h. 0^m. 0^s$; when the first point of Aries is 15° west of the meridian, the time shewn by the clock is 1^h, and the star which is in the line of sight of the Transit at that instant, has 15° right ascension; a star whose superior transit happens at 18^h, has 270° right ascension; and similarly the clock will shew the right ascensions of all stars that cross the meridian at all subsequent hours, which are thus expressed and registered in sidereal time.

28. Again, by means of the Mural Circle the zenith distances of the heavenly bodies as they cross the meridian, may be observed; and by adding the observed meridian

zenith distance ZM (fig. 2), to the known declination QM of a star which crosses the meridian a little south of the zenith, the latitude QZ of the place is obtained; and this being once* accurately known, the declinations of all stars may be determined by taking the sum or difference of the latitude and observed meridian zenith distance; the sum, $QZ + Zg = Qg$ when the passage is between the zenith and pole, and the difference in other cases; and thus a catalogue may be formed of the places of the stars relative to the particular positions occupied by the first point of Aries and the equator at the time of the observations; for it will be seen hereafter that, whilst the stars preserve their respective distances from one another, the situations of the assumed point and great circle from which right ascension and declination are measured are not invariable, but change by a slow continuous motion.

First observations of the Sun's proper motion.

29. Besides the diurnal motion already spoken of, common to all the heavenly bodies, some of them have *proper* motions, and are seen to change their place with respect to the fixed stars; the most remarkable of these is the Sun. The roughest observations shew that the points of the horizon where the Sun rises and sets, and his meridian altitude, are liable to perpetual changes; these, which have no existence in the case of the fixed stars, can only arise from changes in the Sun's declination.

Also his right ascension alters; for if a star be situated a little to the east of the Sun, it will be seen to set after him; soon it will be visible neither at its setting nor rising, being overpowered by the Sun's light; afterwards it will be seen to rise before the Sun; this shews, since the star's position in the heavens is invariable, that the Sun must have moved from west to east till he has passed the star. Moreover if the Sun were stationary, the same constellations would always be seen, contrary to experience, in the south at the same hour after sun-set.

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* The distances of the stars from the pole of the small circle described by the Pole-star (i. e. the pole of the heavens) may be determined, if the latitude be only approximately known for the purpose of correcting for refraction.

30. If we apply the test of instrumental observation, we further discover not only that these motions of the Sun in declination and right ascension exist, but that they are *not equable.*

The circle shews that the Sun's meridian zenith distance remains nearly the same for successive transits, only at two seasons, viz. about the 21st day of June and December, when it has respectively its least and greatest value; in the former case the Sun being a certain distance north of the equator, and in the latter, an equal distance south; at other times the meridian zenith distance perpetually alters, the daily change being the greater, the nearer the Sun is to the equator, in which he is found on the 21st day of September and March.

Again, the Clock and Transit shew that if a star and the Sun cross the meridian at the same instant on any day, at the following passage the star comes first, and the Sun after an interval of variable duration depending upon the time of year, the average value of which is about 4^m, corresponding to something less than 1^0 of right ascension, which must have been described by the Sun from west to east. On the second day the interval between the transits of the star and Sun will be about 8^m, and it will go on accumulating by variable increments, till after $365\frac{1}{4}$ revolutions of the heavens, it becomes 24^h, or the Sun is again in the declination circle passing through the star, and has performed the circuit of the heavens. This period, called a *year*, in which the Sun returns to the same position among the fixed stars from which he started, forms one of the most important divisions of time.

Ecliptic defined. Mode of ascertaining a body's place relative to it.

31. The comparison of the observed declinations and right ascensions of the Sun's center shews that the path which the Sun's center describes among the fixed stars lies all in one plane passing through the center of the Earth; this plane, as well as the great circle according to which it cuts the celestial sphere, is called the *Ecliptic.*

The angle of inclination of the ecliptic and equator is called the *Obliquity of the Ecliptic*, its value being $23^0 . 28'$;

and the line in which their planes intersect, meets the celestial sphere in two points called the *Vernal* and *Autumnal Equinox.* The ecliptic is divided into twelve equal parts or *Signs*, each consisting of 30°, and denoted in order thus; Aries (♈), Taurus (♉), Gemini (♊), Cancer (♋), Leo (♌), Virgo (♍), Libra (♎), Scorpio (♏), Sagittarius (♐), Capricornus (♑), Aquarius (♒), Pisces (♓).

R 32. The *First point of Aries* is the Vernal Equinox, where the Sun's center appears (relative to northern latitudes at the beginning of spring) when he passes from the south to the north side of the equator; it is the fixed point of the equator noticed above, from which right ascension is measured, from west to east, or according to *the order of the signs.* To find the time of the Sun's being in the equinox, we have only to observe his meridian zenith distances on two successive days, one greater, and the other less, than the latitude; then between the observations he was in the equator, and how long after the first observation that happened, may be determined by a proportion, because the change of declination may be supposed to be uniform during the interval between the observations.

R 33. The *Planets* or wandering stars, which are distinguished from the fixed stars by not preserving permanent situations relative to the latter, and by their apparent magnitudes, were supposed, prior to recent discoveries, to be confined to a space of 8° on each side of the ecliptic, which was called the *Zodiac;* on this account the divisions of the ecliptic were called *Signs of the Zodiac.*

The Moon's apparent path among the fixed stars is also nearly a great circle, inclined to the ecliptic at an angle of about 5°; which she completes, by a motion from west to east, in 27^d. 7^h, returning nearly to the same point.

R 34. The *Solstitial Colure* is a great circle passing through the poles of the equator and ecliptic, and intersecting the ecliptic in two points called the *Solstices.* Since this circle passes through the poles of the equator and ecliptic, it is perpendicular to each of them, and, therefore, to the line of their intersection, and consequently cuts them at 90° from the equinoxes. The solstice which lies north of the equator, and in which the Sun appears at Midsummer, is called (relative to places in north latitude) the *Summer* Solstice: the other, in which the Sun is at the beginning of Winter, is called the *Winter* Solstice. The *Equinoctial Colure* is a declination circle passing through the equinoxes.

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35. The *Latitude* of a star is its angular distance from the ecliptic, measured on the arc of a secondary to the ecliptic intercepted between the star and ecliptic; and is reckoned north and south from 0^{0} to 90^{0}. The Sun moving in the Ecliptic has no Latitude.

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The *Longitude* of a star is the arc of the ecliptic intercepted between the first point of Aries and a secondary to the ecliptic passing through the star; and is measured, like right ascension, from west to east through 360^{0}.

The latitude and longitude determine the place of a star with respect to the ecliptic, in the same manner as right ascension and declination determine a star's place with respect to the equator; they are not subjects of immediate observation, but are deduced by computation from the observed right ascension and declination. The latitude and longitude of a star must be carefully distinguished from the latitude and longitude of a place on the Earth's surface; they are equally spherical co-ordinates, designed to fix the positions of points under consideration; but the planes to which they are referred are different.

Longitude is often expressed by the number of signs, i. e. of spaces of 30^{0}, which it contains; thus $187^{0}.19'$ is written $6^{s}.7^{0}.19'$; sometimes the symbol for the sign, in which the remaining odd degrees, minutes, and seconds, are to be taken, is substituted for the number of signs, thus the above may be written ♎ $7^{0}.19'$.

The angle contained between the circles of declination and latitude passing through a star, is called its *Angle of Position.*

Combination of the Sun's proper motion with the general motion of the Heavens. Variation in the length of the Solar day.

36. We will now trace the changes in the position of the Sun's diurnal circle, as observed at a given place, in the course

of a year. Let fig. 6, represent the eastern hemisphere of the feigned celestial vault, the dotted lines being situated in the western hemisphere. Z is the zenith and HR the horizon of any place on the Earth's surface having north latitude; QE♈ the equator, and N♈M the ecliptic, intersecting one another in the line ♈♎. The points ♈ and ♎ are the equinoxes, ♈ being the *vernal* equinox; for the Sun in passing through it, since he moves in the ecliptic from west to east, passes to the north side of the equator, as the figure shews, which agrees with the definition of the vernal equinox; the Sun is in ♈ in spring, he then rises in the east point, his diurnal circle is the equator, and the day and night are of equal duration. A few days afterwards, his place in the ecliptic will be S; then this point and the Sun, by the revolution of the sphere, will be transferred to the meridian at C, that is, his circle of daily motion will be SOC, O being the place where he rises, and ZC his meridian zenith distance; and after another interval of a few days, his place will be S' and his diurnal circle $S'C'$. Advancing in this manner each day along the ecliptic, his diurnal circle will recede more and more from the equator, and his meridian altitude and the length of the day will increase; till, having described 90^0 of the ecliptic, the Sun comes to the summer solstice M, which is the limit of his approach to the north pole; his circle of daily motion is then the parallel BD which touches the ecliptic in M, and is called the Tropic of Cancer, and the day is then the longest. After passing the solstice, the Sun begins to return to the equator, appearing, by the revolution of the heavens, to describe the same series of diurnal circles as before. When he reaches ♎ the autumnal equinox, he is again in the equator; he then passes to the south side of the equator, and describing 90^0 of the ecliptic, comes to N the winter solstice; when, having attained his greatest south declination, the length of the day is the shortest, and his diurnal circle is FG, the Tropic of Capricorn. The Sun then begins to return to the equator through N♈, and the days lengthen; till at the end of a year, or after about $365\frac{1}{4}$ revolutions of the heavens, he is again in the first point of Aries.

This combination of the Sun's proper motion with the daily motion of the heavens, makes his actual path a sort of spiral traced on the celestial vault; but so small is the proportion which his proper motion bears to the general diurnal motion, that at present it is sufficient to conceive that every day, from his rising to his setting, the Sun describes a circle like a fixed star, parallel to the equator, and distant therefrom, by his own distance when on the meridian; but this circle every day changes its situation, approaching to and receding from the equator, on both sides, within certain limits.

37. Next to trace the changes in the length of the solar day at different places on the Earth's surface in the course of a year. In the case of the fixed stars we have seen (Art. 20.) that the nearer any one of them is to the visible pole, the greater is the portion of its diurnal circle which is above the horizon, and the longer consequently does the star continue above the horizon. Hence at all places having north latitude, from the winter to the summer solstice, since the Sun's north polar distance when on the meridian is less each succeeding day, a greater portion of his diurnal circle lies above the horizon each succeeding day; or the length of the day increases, and that of the night decreases; the Sun's being above or below the horizon being understood to constitute the distinction between day and night. At places having the south pole above their horizon, the reverse takes place; the length of the day decreases from the southern to the northern solstice, and increases from the northern to the southern solstice; at the equinoxes the days and nights are of the same length at all places on the Earth's surface.

Also the length of the day at any place when the Sun has north declination is equal to the length of the night when he has the same south declination; for if ML, GF (fig. 2) be the Sun's diurnal circles at those times, it is manifest that $RV = Hv$, and therefore the chords DO, do, being perpendicular to HR, are equal; consequently $DMO = oFd$, and the times of describing them are equal; hence in the course of a year, the whole period of day-light is very nearly the same at every place on the Earth's surface.

38. Through N, fig. 7, the winter solstice, draw NQ the arc of a great circle perpendicular to the ecliptic; then ΥN

being 90^0, the point ♈ is the pole of NQ, which therefore being produced passes through P the pole of the equator; and if $NK = 90^0$, K is the pole of the ecliptic, and the great circle $PKNM$ is the solstitial colure. Also, since the arc of a great circle which joins the poles of two great circles measures their inclination, $PK = NQ = \angle N$♈Q is the obliquity of the ecliptic, or inclination of the ecliptic and equator. The great circle P♈A is the equinoctial colure.

Through S any star draw secondaries to the ecliptic and equator KS, PS, meeting them respectively in O and D; then ♈O is the longitude, SO the latitude, and $\angle KSP$ the angle of position of the star S.

If we suppose O to be the place of the Sun in the ecliptic, the figure evidently represents the position of the great circles of the sphere, with respect to the meridian and horizon of the place whose zenith is Z, a short time before sunrise on some day between the ends of March and June. The diurnal motion of the heavens will cause K, the pole of the ecliptic, to describe a small circle TK, distant $23^0.28'$ from the pole of the heavens, which is called the *arctic* circle; the circle described at the same distance about the opposite pole is called the *antarctic* circle.

Rotation of the Earth round an axis.

39. It has been said that the stars appear to describe parallel circles round the axis of the heavens in $23^h.56^m$, but it is evident that the same effect will be produced if the stars remain at rest, and the Earth revolve uniformly round the same axis in the same time, but in the opposite direction. For the spectator, not being aware of his own motion, would refer the whole change of relative position to the stars, which would appear to describe parallel circles in a direction opposite to his own motion. That *the Earth really revolves* is rendered R
probable; 1st, by its greater simplicity, when we consider the immense distance of the stars, and the enormous rapidity with which they must revolve, in order to move round the Earth in $23^h.56^m$; 2d, from the analogy of the Sun and planets, which, although many of them are much larger than the Earth, are found by observing spots on their disks to have such a rotation; 3d, from the Earth's figure being not exactly spherical,

but nearly such as it would be, supposing the Earth to have been originally fluid, and to have revolved round an axis; and from the diminution of gravity in proceeding towards the equator being such as would arise from the centrifugal force of the Earth's rotation.

It has been tried to determine the question experimentally, from considering that if the Earth revolve round its axis, a stone let fall from the top of a tower, ought not to fall exactly at the foot, but a little to the east of it; for the stone, at the moment of falling, has the horizontal velocity of the top of the tower, and, as this is greater than that of the places it successively arrives at, it ought to have a relative easterly motion from the tower, and ought, therefore, to fall east of the foot of the tower. The experiments* appear to have been tolerably decisive in favor of the hypothesis of the Earth's rotation, notwithstanding the difficulty of measuring accurately the small deflection from the vertical, and getting quit of all extraneous causes that may affect its amount.

Motion of the Earth round the Sun.

40. The same difficulty of distinguishing between real and apparent motion occurs with regard to the Sun. By observing his successive positions relative to the fixed stars, we have seen that the Sun seems to move through the heavens in a direction opposite to the apparent diurnal motion, or from west to east; but the Sun would appear to move in the same way if he remained at rest, and the Earth moved round him in the same direction and period. For, let S (fig. 8) be the Sun, E, E' the Earth, in any two points of its orbit round S; through E' draw $E's$ parallel and equal to ES. Then, when the Earth was at E, the Sun appeared among the fixed stars in ES produced, or, on account of their immense distance, in $E's$ produced; and when the Earth is at E', the Sun is seen in the direction $E'S$; therefore, he has appeared to describe the angle $SE's$ round the Earth, equal to ESE' the angle described by the Earth round the

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* See Reich, *Fallversuche über die umdrehung der erde* Freiberg 1832. In a vertical descent of 158.5111 metres, the observed mean deviation was 28.396 millimetres; theory gives 27.512 millimetres.

Sun. The Sun's proper motion lying in a plane passing through the Earth's center, his apparent path on the surface of the sphere is circular, and depends only on the plane in which he moves, and not on the actual curve traced in that plane; but since when the Earth was at E, the distance of the Sun was $ES = E's$, and that the angle $ESE' = SE's$, therefore, the Sun must appear to describe the same curve in space round the Earth which the Earth describes round the Sun; and whatever variations there may exist in the velocity with which the Earth describes her orbit, the same will affect the Sun's motion in his apparent orbit.

41. There is reason to suppose that *the Earth really moves round the Sun*. For, from the distance of the Earth from the Sun and the period of the Sun's apparent revolution round the Earth, compared with the distances and periods of the planets which we know revolve round the Sun, it seems probable that the Earth is a planet, and is subject to the same laws of motion as the others. The aberration of light also, and the stationary points of the superior planets, (which will be treated of hereafter) can be simply explained on the hypothesis of the Earth's motion round the Sun, but have not been accounted for on the contrary supposition. But the strongest proof of the hypothesis of the Earth's double movement, is the perfect agreement which it establishes between observations and results; at this stage, it can only be shewn to be far more probable than the contrary supposition; in the sequel of the work, it will appear verified by more proofs than are united in the support of a theory in any branch of Natural Philosophy. We conclude, therefore, that the Earth revolves round its axis in a day, and round the Sun in a year; but as the general phenomena would be the same on the contrary hypothesis, we may, in explaining them, make use of that supposition which is most convenient.

True statement of the heavenly movements, and consequent explanation of the phenomena.

42. The true statement then of the heavenly movements is, that the center of the Earth describes about the Sun S, (fig. 9) fixed in space, an orbit $Tt'T't$ in one plane, which is nearly

a circle with the Sun in its center, in a period of $365\frac{1}{4}$ days, moving from west to east, whilst at the same time the Earth revolves about its axis once each sidereal day in the same direction; this axis is carried forward in space, preserving its parallelism in all positions, and making a nearly constant angle of $66^0.32'$ with the plane of the ecliptic.

And it will now be useful to shew that, proceeding upon this true statement, we can readily explain those grand phenomena previously accounted for on the supposition of the Earth's quiescence; viz. the general parallel motion of the Sun and all heavenly bodies, and the Sun's proper motion in declination and right ascension.

43. To begin with the apparent daily motion of the Sun and the other heavenly bodies, by the effect of the Earth's diurnal rotation, the different portions of its surface being successively presented to the Sun and turned from him, are made to form parts of the illuminated and unilluminated hemisphere, and the alternations of day and night are produced. Also when the plane of the horizon of any place is sufficiently turned away from the Sun, the stars of half the celestial vault become visible; and the Earth turning from west to east, the horizon gradually *uncovers* stars in the eastern portion of the heavens, and hides others in the western portion; and thus, during a winter's night, nearly the whole heaven is presented to our view, except two portions, one near the south pole which never rises, and another contiguous to the Sun's place in the ecliptic at that time, which being above the horizon only at the same time as the Sun, is concealed by the glare of day.

44. Again in consequence of the Earth's annual motion, the Sun seems to traverse the ecliptic in the same direction as the Earth actually describes it; when the Earth is at T, the Sun appears to occupy, relative to the fixed stars, the point where the firmament is penetrated by the line TS; and in whatever point of the ecliptic the Earth is, the Sun appears in the opposite point, separated from the Earth by 180^0 of longitude, and in the position which the Earth will really occupy after six months.

45. It has been stated that the angular distance of two fixed stars is the same, seen from all points of the Earth's surface; it is further very doubtful whether the slightest alteration can be detected in it, when viewed from all points of the Earth's orbit; hence not only is the Earth's diameter, but the diameter of the Earth's orbit evanescent, compared with the distance of the fixed stars. This removes all difficulty in reconciling the assertion respecting the invariable position, with respect to the fixed stars, of the poles and equator, with the actual state of the case which is, that the Earth's axis, being carried forward parallel to itself round the orbit, describes a cylindrical surface, and being produced, its extremity traces out a curve on the feigned celestial vault; and that the equator, retaining its parallelism, has its extreme positions widely separated. On account of the prodigious distance of the fixed stars, the different parallel positions of the Earth's axis and equator, being produced, cut the celestial vault in invariable points, and in an invariable circle, just the same as if the Earth were fixed.

Explanation of the apparent changes of the Sun's declination.

46. We shall next shew that the regular periodical changes of the Sun's declination, and consequently the vicissitudes of the seasons, are owing to the Earth's annual motion, whilst its axis preserves a constant inclination to the plane of its orbit, and is always parallel to itself. Let S fig. 9, be the Sun in the center of the orbit, and T the Earth's center; let the radius vector ST meet the terrestrial sphere in N; a plane perpendicular to the Earth's axis through N will give the small circle NB, all the points of which, in consequence of the Earth's rotation, will in succession pass through N; and the Sun being fixed in S, his diurnal path to a spectator placed in T would be the intersection of the conical surface NTB with the celestial vault. But a spectator any where on the Earth's surface views the celestial movements nearly the same as if he were at the center; therefore, for the purpose of general explanation, we may say that when the Earth is at T, at all places on its surface, the Sun's apparent diurnal path is the intersection with the celestial vault of the conical surface NTB, and his polar distance is the angle PTS

Suppose the Earth at *T* to have that position for which the plane containing the Sun and Earth's axis is perpendicular to the plane of the ecliptic, this will happen at the summer solstice. The circle which the Sun then appears to describe in 24^h, is the tropic of Cancer distant $23^0.28'$ north of the equator, which name is also given to the parallel *NB*, similarly situated with respect to the Earth's equator. The inhabitants of the parallel *NB* have the Sun *vertical* or in their zeniths; those of the zone *NPB* do not, it is true, see the Sun in their zeniths, but he then makes his nearest approach to their zeniths. A plane *UV* through *T* perpendicular to *ST* separates the enlightened and unenlightened hemispheres; it cuts the meridian *PBA* $23^0.28'$ from *P*, and consequently all places less distant from *P* will never, by the diurnal rotation, be carried out of sight of the Sun, and an equal portion at the opposite pole will never be brought into sight of him, when the Earth is in this position; and the nearer any place is to *P*, the smaller portion of its diurnal path will lie on the dark side of the plane *UV*.

When the Earth, after leaving *T*, comes to the opposite point T' of its orbit, and the axis $P'A'$ is again in the plane of the solstitial colure, since it preserves its parallelism, the angle $P'T'N'$ or the Sun's north polar distance is obtuse, and supplementary to its former value; consequently the radius vector meets the Earth's surface in a point whose declination is $23^0.28'$ south; the Sun appears to describe the tropic of Capricorn in 24^h; the inhabitants of the zone $N'A'B'$ have summer, whilst in the opposite hemisphere the epoch is the winter solstice; the south pole is now turned towards the Sun, and at parts near the north pole there is perpetual night.

In intermediate positions, when for instance the Earth's center, after leaving *T*, moves towards t', the angle *STP* which is the Sun's polar distance goes on increasing till it becomes a right angle at t'; the radius vector then meets the Earth in the equator at q', and the Sun consequently appears to describe the equator, the epoch being the autumnal equinox; also the plane which separates the enlightened and unenlightened hemispheres then passing through the poles, the Sun is visible and invisible alternately, at every part of the Earth's surface, for exactly half the time of her rotation.

From t' to T' the angle $St'p'$ is obtuse, or the Sun's diurnal circle is south of the equator, and his north polar distance goes on increasing, till at T' it has its greatest value; and through the remainder of the orbit it diminishes, becoming a right angle when the Earth, at the vernal equinox, is at t, and the Sun is again visible every where, and for the same time; after which the Sun's diurnal circle passes to the north of the equator.

47. Thus it appears that the parallelism of the inclined axis, about which the Earth rotates, makes us suppose that the Sun's declination changes, and that he describes a series of parallels to the equator, moving backwards and forwards between the tropics; each of these parallels is the effect of our daily rotation, and the passage from one to another, or the change of declination, is due to the translation of the Earth in her orbit.

The appearances would be different if the Earth's axis were either perpendicular to the ecliptic, or coincident with it at the same time remaining always parallel to itself; in the former case, the Sun would never have declination but always be seen in the equator, and there would be perpetual spring, and days and nights of equal lengths every where; in the latter, his declination would admit of all degrees of magnitude north and south, and he would be seen alternately in each of the poles.

48. As the Sun seems to pass to the north side of the equator when the Earth is at t, his place in the ecliptic (which is that great circle according to which the plane of the Earth's orbit meets the celestial vault) determined by producing tS to meet the firmament, is ♈ the first point of Aries, and the Earth's place is consequently ♎ the first point of Libra. The line ♈ ♎ demands particular attention, as it is often referred to; it coincides, as we perceive, with the intersection of the Earth's equator and plane of her orbit, when the Earth is at t and t'; but on the supposition that the Earth's axis preserves its parallelism, the line of intersection likewise preserves its parallelism; consequently ♈ ♎ is the line drawn in the plane of the ecliptic to which the line of intersec-

tion of the Earth's equator with that plane continues parallel, as the Earth describes her orbit. Again, when the Earth is at *T*, the Sun is seen to have his greatest north declination, and his place in the ecliptic, determined by producing *TS*, is ♋ the first point of the sign Cancer, and the Earth's place is ♑ the first point of Capricorn.

49. The Sun's longitude is the arc of the ecliptic between his place in the ecliptic and the first point of Aries, measured from west to east, that is, in the direction in which the Earth moves. Consequently when the Earth's center is in the positions t_1, t_2, the Sun's longitude will be the angles ♈ $t_1 S$, ♈ $t_2 S$ respectively, (the latter being greater than 180^0); for lines drawn from the Earth in all positions to the first point of Aries, are parallel to one another. Also the Earth's longitudes, seen from the Sun, will be ♈ St_1, ♈ St_2, in each case differing from the corresponding longitude of the Sun by 180^0.

Explanation of the apparent changes of the Sun's right ascension.

50. We next proceed to shew how the apparent daily change of the Sun's right ascension is produced by the Earth's motion in the ecliptic. Let at any time the Earth's center be in the point *T* of its orbit, (fig. 10) and let the Sun be on the meridian of a place *M*; then he will appear in the heavens contiguous to some star *s*, which is on the meridian at the same time with him. In 24 sidereal hours let the Earth's center arrive at *t*, and let the radius vector *St* meet the Earth's surface in the meridian *pb*; then the meridian of *M* will have performed a revolution, and be parallel to its former position, and therefore, on account of the immense distance of the fixed stars, will again pass through the same star *s*; but will not have reached the Sun, which at that instant is on the meridian *pb* a little to the east of *M*. Consequently it will not be till some minutes after passing the star *s*, that the meridian *pM*, by the revolution of the Earth (which takes place in the direction *aq*, the same as the direction *Tt* of the motion of its center) will be made to pass through the Sun; the angle *Mpq* which the meridian describes in that interval, never differing much from *stS* or *TSt*, the angle described by the Earth's

center round the Sun in the same time. This then is the true explanation of what is observed, namely, that if a star and the Sun leave the meridian together, the following transit happens about four minutes later for the Sun, than for the star.

51. Hence also we perceive not only that the solar day exceeds the sidereal day, but that the excess is variable; for the angle apq which measures the excess would not remain the same, supposing btg not to alter, by reason of the change of Stp the Sun's polar distance; and there is, as we shall see, a second cause of inequality in $btg = TSt$, the angle described by the Earth in a sidereal day, being variable.

Zones of the Earth's surface. Comparison of the phenomena presented by the Sun at places in different Zones.

52. The small circles of the terrestrial sphere parallel to the equator and distant $23^0 . 28'$ north and south of it, as has been stated, are called the Tropic of Cancer and Capricorn; and the portion of the Earth's surface which lies between them is called the Torrid or intertropical zone. Also the small circles distant $23^0 . 28'$ from the north and south pole of the Earth are called, respectively, the Arctic and Antarctic circle, and the portions of the Earth's surface which lie within them, the Arctic and Antarctic zone. The remaining portions of the Earth's surface situated on each side of the equator, between a tropic and a polar circle, are called respectively the north and south Temperate zone.

The grand distinctions in the phenomena observed in these divisions of the Earth's surface, are that at places between the tropics the Sun is vertical or passes through their zeniths twice in the year; at places within the polar circles, during a certain portion of every year, the Sun never sets, and during another equal portion he never rises; whilst at places in the temperate zones he is never vertical, but always crosses the meridian on the same side of the zenith, and is below the horizon a portion of every 24 hours.

53. To trace the changes in the phenomena offered by the Sun in the course of a year to spectators in different zones.

To the inhabitants of the equator, the poles are in the horizon, the diurnal circles of the Sun and stars are all perpendicular to the horizon and bisected by it, all the year round the days and nights are equal, at each of the equinoxes the Sun is vertical, at the solstices his meridian zenith distances are each equal to 23°. 28′, and in opposite directions; the shadows at noon fall during one half of the year towards the south, and during the other half towards the north, and the year consists of two summers and two winters. At all places throughout the torrid zone the phenomena are analogous, the circles of daily motion make an angle greater than 66°. 32′ with the horizon, and the Sun crosses the meridian, part of the year to the south, and part, to the north of the zenith. In the temperate zones, since the Sun never attains the zenith of any place, the shadows at noon always fall towards the north, or towards the south. At places situated on the arctic circle the Sun never sets on the longest day; for his north polar distance is then 66°. 32′, which is equal to the latitude or altitude of the pole at those places; hence at his inferior transit the Sun will just touch the horizon. At places within the arctic circle, since the elevation of the pole exceeds 66°. 32′, the limit to which the Sun's north polar distance diminishes, therefore for a certain number of days before and after the summer solstice, the Sun's diurnal circle will be wholly above the horizon, i. e. he will never set, and in the same way for an equal period at the winter solstice, he will never rise; and at the pole he will be half the year above, and half below, the horizon.

The Sun's influence on the climates of different places. The trade winds.

54. The Sun's influence in producing warmth, depends upon the *time* during which he is above the horizon, and the *elevation* he attains above the horizon; for the greater length of the day gives him a longer time to act; and the greater his altitude, the greater proportion of his rays does a given horizontal surface receive (for the illumination of a plane varies as the sine of inclination of the plane to the direction of the incident rays), and the more directly do they fall upon the atmosphere, and consequently, having a shorter distance to traverse through it, a smaller quantity of the solar light and heat is absorbed. At the equator since the Sun is always twelve hours above the horizon, the greatest heat will happen

at the equinoxes when he is vertical, and the least at the solstices. In the temperate zones, the meridian altitude of the Sun and his stay above the horizon continually increase from the winter solstice to the summer solstice (relative to the place), which are consequently the periods of least and greatest heat; or rather it is *after* midsummer that the period of greatest heat occurs, for though the actual quantity of heat then received is less, yet the Earth's surface and atmosphere receiving more heat than they part with, the general stock is increased. In the torrid zone, the two causes of solar power conspire only from the winter solstice (relative to the place) till the Sun becomes vertical, afterwards they are opposed, the elevation diminishing whilst the length of the day increases, till the next solstice.

If when the Sun is south of the equator, a person in north latitude approach the equator, the Sun's meridian altitude above his new horizon and the length of the day are both increased, and therefore the causes conspire to make the winter milder, as the north latitude diminishes. But if when the Sun is north of the equator, a person move from a place, whose zenith is north of the Sun, towards the pole, the length of the day will be increased, but the meridian altitude above the new horizon diminished, i. e. the causes of the Sun's power are opposed. The above is a general explanation of the way in which the Sun's position affects the climates of different places; but so great is the influence of local causes, that frequently the climates of places having the same latitude are very different.

55. The solar heat is most effective at the equator, and by rarefying the air there, causes it to move in the direction in which it is least opposed, i. e. to ascend and expand itself along the upper strata of the atmosphere towards the poles, whilst its place is supplied by the cooler and denser parts rushing along the surface in the contrary direction. But the atmosphere partakes of the velocity of rotation of that part of the Earth's surface with which it remains in contact for any time, and that velocity, as we know, increases as the latitude diminishes; consequently the aerial current, rushing towards the equator, revolves more slowly than the parts of the Earth at which it successively arrives, and bodies on the

Earth's surface will strike against it with the excess of their velocity, and be opposed by its reaction. Hence to a person between the tropics, the combination of the relative easterly motion, and the absolute motion towards the equator, of the aerial current, will produce the sensation of a wind from the north east, or south east, according as he is in the northern or southern hemisphere; at the equator, the wind will blow from the east, which is the direction of the Trade winds. These results are verified by experience, and so far confirm the hypothesis of the Earth's rotation.

Variation of the Sun's apparent diameter. His apparent orbit an ellipse with the Earth in the focus.

56. Hitherto we have said nothing of the Sun's distance from the Earth, either as to its actual magnitude or its variations; all we know is, that he moves always in the same direction in a plane through the Earth's center, and that consequently, whatever be his path in that plane, when referred to the celestial vault, he appears to describe a great circle. It can be readily shewn that his distance is continually varying, and that his actual path is an ellipse. Through the Sun's center, perpendicular to the line joining that point and the eye of the spectator, conceive a plane to be drawn; this will determine the visible hemisphere of the Sun, which appears like a flat disk; and if lines be drawn from the extremities of any diameter of the disk to the eye, the angle between them, i. e. the angle subtended by the Sun's real diameter at the eye of the spectator, is called his *apparent diameter;* and since this angle is very small, the real diameter may be considered as a circular arc concentric with the spectator's eye, in which case we have, distance × (apparent diameter) = real diameter, which shews, since we have no reason to suppose that the real diameter alters, that the apparent diameter varies *inversely as the distance.* Now the apparent diameter can be accurately measured, and is found to be continually varying, therefore the Sun's distance is continually varying; its mean value, as will be seen, being about 26 thousand times the Earth's radius.

57. Also the apparent diameters at different times give the proportion of the corresponding distances of the Sun;

and the right ascensions and declinations being observed, give his corresponding longitudes; hence, knowing the directions of the radii vectores and the lengths of lines which are proportional to their respective lengths, we can trace out a curve similar to that described by the Sun round the Earth in the plane of the ecliptic; and this is found to be *an ellipse with the Earth in its focus.*

In an ellipse, the longest and shortest lines that can be drawn from the focus to the curve, are the two portions of the major axis; and accordingly we find that the Sun's positions, when his apparent diameter is least and greatest, are separated by 180^{0} of longitude; and that the apparent diameter continually increases from one position to the other, and then diminishes through the remaining half of the orbit.

58. The least apparent diameter is nearly $31\frac{1}{2}'$, and is at present observed about the 2nd of July, and the greatest $32'.35''$, observed about the first of January, at which times the Sun's distance from the Earth is respectively greatest and least, and the corresponding points of his orbit are called the *apogee* and *perigee.* The extreme variations of the apparent diameter being so small, the solar ellipse is nearly a circle; the greatest distance diminished by about one thirtieth of itself gives the least distance. Conforming our language to appearances, we have attributed to the Sun the motion and variations of distance which really belong to the Earth; this the reader will rectify for himself, both here and in the following Article; when the Earth is considered the moving body, the points in her orbit most remote from and nearest to the Sun are called the *aphelion* and *perihelion,* or the *apsides,* and the line joining them, the *line of apsides.*

Equable description of areas by the Sun's radius vector.

59. Having thus ascertained that the Sun appears to move in an ellipse round the Earth, the next enquiry that suggests itself, is whether his velocity in that ellipse is uniform or variable. By finding the Sun's longitudes in two consecutive positions, from the observed right ascensions and declinations, and taking their difference, we obtain the angle described by the Sun round the Earth in the time between

the observations; and dividing that angle by the time, we get a quantity which expresses the angular velocity at the first observation, the more nearly, the smaller the time and angle are. Now this angular velocity is found to alter with the Sun's distance, so as to be greatest when the apparent diameter is greatest, and least when the apparent diameter is least; but yet to exhibit *greater* inequalities than those of the apparent diameter or distance, and not the *same*, which would be the case if the Sun moved uniformly in his nearly circular orbit. On accurate comparison of the angular velocities with the distances, it is found that they vary *inversely as the squares of the distances;* consequently, if the square of the radius vector be multiplied by the angle described by it in any the same small interval, as an hour, this product will be the same for every point in the orbit. But half of this product expresses the area of a circular sector whose radius is the first distance, and angle the small angle in question, an area which (since the Sun's orbit is nearly a circle) is undistinguishable from that contained by the two distances of the Sun and the intercepted arc of the orbit; hence the sectorial area described by the radius vector in any small interval, as an hour, is independent of the Sun's distance, the change produced in it at different points by the increase or diminution of the distance, being exactly compensated by an opposite change in the angle. But if the areas described by the radius vector in small equal portions of time be equal in every part of the orbit, the whole areas described by the radius vector in equal portions of time of any magnitude will also be equal. Consequently the law which regulates the velocity of the Sun in his orbit is, that *the area described by the radius vector in any time is proportional to the time.*

60. At perigee the arc of the ecliptic described by the Sun in 24^{h}, is $61',165$, at apogee it is $57',192$; hence the greatest difference between the angular velocities is about a fifteenth part of the least angular velocity. If we divide 360^{0} by $365^{d}.\,5^{h}.\,48^{m}.\,49^{s}$, which is the length of a tropical year, we get $59'.\,8''\frac{1}{3}$ for the *mean* length of the arc of the ecliptic or equator described each day by the Sun, that is, his mean daily motion in longitude or right ascension.

Different lengths of the Seasons produced by the Sun's elliptic motion.

61. Let *KMN*, fig. 11, be the solstitial colure, ♈ *MN* the ecliptic, *A* the place of apogee referred to the ecliptic near *M* the summer solstice, *P* that of perigee near *N* the winter solstice, and consequently *AP* the direction of the major axis or line of apsides of the solar ellipse *anp*, passing through *t*, the Earth, in its focus. Then because the Sun's angular motion is slower the further he is from the Earth, (Art. 59) he will take a longer time to describe the 180^0 of longitude, 90^0 of which are on each side of apogee, than the remaining 180^0 of his course, in which he passes through perigee; and accordingly we find that he is longer on the north than on the south side of the equator, by a period amounting at present to about $7^d.16^h.50^m$.

At apogee the daily motion in longitude is least, and continues the same for several days together, and this happening at present near the summer solstice when the motion in declination is hardly perceptible, it follows that shortly after midsummer the variation in the length of the day will on both accounts be less than at any other season, and there will consequently occur several successive days nearly of the same length; the same thing happens at the winter solstice but in a smaller degree, for the circumstance of the Sun being then in perigee and moving quickest, soon causes the change in declination to sensibly affect the length of the day.

62. Hence we may compare the lengths of the four seasons. Of the four sectors into which the solar ellipse is divided by the lines *MN*, ♈ ♎, at right angles to one another, that will be the greatest whose bounding radii are greatest, viz. *mtx*, and the opposite one *ntv* will be the least; consequently the quarter of the year from the northern solstice to the equinox is the longest, and the quarter from the southern solstice to the equinox the shortest, and that from the equinox to the northern solstice is longer than the remaining one. Hence if the periods during which the Sun's north declination increases and diminishes be called respectively *spring* and *summer*, and the periods during which his south declination increases and diminishes *autumn* and *winter*, the order

of these periods in respect of length is at present (for we shall see by the ~~next~~ 68th Article that it is not permanent) summer, spring, autumn, winter.

Hence also the Sun's influence in communicating light and heat to the Earth is greatest, all other things continuing the same, at the winter solstice; but the other circumstances on which it depends are so much more powerful in modifying and increasing it, that the variation of the Sun's distance may in comparison be neglected.

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Modes of measuring time. Different sorts of days. Equation of time.

63. We are now enabled to explain the mode of measuring *time*, which is a matter of great importance, and not without difficulty; and it may be proper to repeat the following definitions. A *day* is the interval of time between the departure of any meridian from a heavenly body and its succeeding return to it and derives its name from the body with which the motion of the meridian is compared. The interval between the departure and return of a meridian to the Sun's center is called a *Solar* day; in the case of the Moon, the interval is called a *Lunar* day; and in that of a star a *Sidereal* day. *Apparent* noon at any place is the moment when the Sun's center is actually upon the meridian at that place.

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64. It has been stated that the diurnal rotation of the heavens is performed in an invariable period of $23^h. 56^m. 3^s,5$ *mean time*, called a sidereal day; and not only is the interval between a star's leaving the meridian and returning to it invariable, but also the times in which equal portions of its course are performed, are equal. However the sidereal day would not be convenient as the ordinary measure of time; its commencement, depending upon the presence of some particular star upon the meridian, would not be marked by a sufficiently striking phenomenon, and in the course of a year would happen when the Sun was at various distances above and below the horizon. That luminary, which regulates the operations of mankind and determines the periods of labour and rest, is pointed out by nature to fix the standard of time.

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65. In common language, the period during which the Sun is above the horizon is called day; but the *solar* or *astro-*

nomical day, as has been said, includes the whole duration of the Sun's diurnal rotation, and is the interval between two consecutive noons or midnights; by reason of the Sun's proper motion from west to east, it exceeds the sidereal day; and that excess is of variable length both on account of the inequality of the Sun's proper motion, and also on account of the obliquity of the ecliptic. For since the arc of the ecliptic described by the Sun in a day, is at one season 57′ and at another 61′, it is clear that even if his declination never altered, solar days would be of different lengths. But if through the extremities of that arc we conceive two meridians to be drawn, the intercepted arc of the equator is the Sun's daily motion referred to the equator, and the time which it takes to be carried across the meridian, is the excess of the solar above the sidereal day; and this arc of the equator manifestly bears a much greater ratio to the corresponding arc of the ecliptic near the solstices when they are parallel, than at the equinoxes when the inclination of the two arcs to one another is greatest. Thus we see that solar days are of unequal lengths; and therefore a clock that goes uniformly cannot keep with the Sun, i. e. its index cannot continue to point 12^h, when the Sun is on the meridian.

The solar day is consequently an unfit measure of time, and is superseded by a duration called a *mean solar day*, which is the computed average length of the actual solar days; i. e. its length is equal to the mean or average of all the apparent solar days in a year; as its commencement never differs much from that of the real solar day, and as its length is invariable, it is free from the objections against either the sidereal or solar day as a standard.

66. The mean solar day may be supposed to be determined by the transits of an imaginary Sun which describes the equator with the Sun's mean motion in right ascension or longitude. If we conceive a star to describe the ecliptic with the Sun's mean motion in longitude, starting with him from perigee, its place marks the Sun's mean place in the ecliptic, and the angle between them is called the *equation of the center;* and the interval between their transits is the difference between mean and apparent time as far as it arises

from the Sun's unequable motion in the ecliptic, and vanishes at apogee and perigee. Again if we suppose the star to be in the equinox at the same instant as the fictitious Sun which, moving in the equator with the Sun's mean motion in longitude, determines by its transits mean noon, they will always be at the same distance from the first point of Aries, and will be on the meridian together at the solstices and equinoxes; the interval between their transits at other times is the difference between apparent and mean time as far as it arises from the obliquity of the ecliptic.

67. The arc of the equator intercepted between two meridians drawn respectively through the centers of the true Sun, and the above mentioned fictitious Sun, reduced to time, and consequently expressing the interval between their transits, R is the *equation of time;* as they are not in the equinox together, nor in the solstitial colure together, the equation of time does not vanish at the equinoxes or solstices. There are however four days in the year at some instant of which the apparent and mean time are the same; these at present are December 24th, April 15th, June 15th, and September 1st; in the interval between the first and second of these days, and in that between the 3rd and 4th, the apparent is always later than the mean time, or the clock before the Sun; between the 2nd and 3rd, and between the 4th and 1st, the clock is after the Sun. The equation of time is computed and registered in tables for every day in the year; apparent time can be determined by a dial, or by observing when the Sun is on the meridian; if to this the corresponding equation be affixed with its proper sign (and consequently be either added or subtracted) we obtain the mean time. The equation of time may amount to $16^{m}.17^{s}$ or $-14^{m}.37^{s}$; i.e. a clock adjusted to mean time may indicate $16^{m}.17^{s}$ or $23^{h}.45^{m}.23^{s}$, when it is apparent noon, i. e. when the true Sun is on the meridian.

Progressive motion of the Sun's Apogee.

68. It is manifest that the different lengths of the seasons, and the variations in the amount of the equation of time, depend upon the relative positions of the apogee and equinoxes; if these are fixed points, or preserve always the same

relative positions, the above results are permanently true; if not, the equation of time for a given day will be different in different years. This leads us to enquire whether the line of apsides and the line of equinoxes retain fixed positions in the plane of the ecliptic, or are in motion; and the conclusions which we arrive at on these points are very remarkable; and they besides exercise great influence upon the division of time into longer periods than days, a subject that will next demand our consideration.

69. The orbit described by the Earth's center in the plane of the ecliptic, does not retain a fixed position, but revolves slowly in that plane about the focus in which the Sun is. This cannot be easily detected by direct observation, because the change of the Sun's apparent diameter takes place too slowly, and is confined between limits too narrow, to enable us to observe the Sun's exact place among the fixed stars when at his least or greatest distance, and so to compare the positions which the line of apsides occupies at different times. But if the Sun be observed in two opposite points of his orbit 180° from one another, and if the time of passing between them be exactly half the time between leaving and returning to the first, those points are the apsides; if that relation does not exactly but very nearly obtain between the times, the line of apsides not exactly but very nearly coincides with the line joining the Sun's places; and from the observations, its exact position relative to the fixed stars can be calculated; and its position having been thus determined at different times, the line of apsides is found to revolve in the plane of the ecliptic from west to east. If the angle described by it in a long interval be divided by that interval (by which means the result is affected with a proportionably small part of the errors attending the determination of the line of apsides in its two positions) the *progressive* annual motion of the apogee, relative to the fixed stars, is found to be $11'',25$.

Retrograde motion upon the Ecliptic of the equinoctial points.

70. In Art. 46, we assumed that the Earth's axis strictly preserved its parallelism, and consequently that the line of

intersection of the ecliptic and equator also preserved its parallelism, and therefore when produced met the celestial sphere in invariable points relative to the fixed stars; but the fact is that at the end of a revolution of the Earth in her orbit, the line of intersection meets the ecliptic in points 50″,2 west of those it occupied at the beginning. This slow *retrograde* motion of the line of intersection of the Earth's equator with the ecliptic, the annual amount of which is so small, does not of course affect the phenomena hitherto described as happening in the course of a year on the hypothesis of its non-existence; but in a long series of years its effects become very sensible and remarkable. We proceed to state how the retrograde motion of the equator on the ecliptic may be detected by observation, whence it arises, and what effect it produces.

71. The Sun's meridian zenith distances, as we have stated, may be observed on two successive days one greater and the other less than the latitude; then between the observations the Sun was in the equator, and at neither of them consequently could the distance of his place in the ecliptic from the first point of Aries exceed 1°; now if the stars which are on the meridian with the Sun the day before he enters the equinox, be noted down from year to year, it is found that they do not continue the same, but are soon replaced by others situated to the west; this at first sight seems to indicate a slow motion of the stars from west to east.

72. Also if catalogues of the stars constructed as explained in Art. 28 for different epochs, be compared, it is found that the places of the stars are differently described in them; the right ascensions and declinations are variously altered, the longitudes are all increased by the same quantity, whilst the latitudes remain the same, and the different stars retain the same positions with respect to each other; the same effect as would be produced if all the stars described circles in planes parallel to the ecliptic, with a common slow angular motion, from west to east, about an axis passing through the poles of the ecliptic. For let *NM*, *EQ*, fig. 11, be the ecliptic and equator, *K*, *P*, their poles, ♈ ♎

their line of intersection; then any star S in the circle of latitude KS, by the common angular motion about Kt in the direction ♈ Q, would be brought into the circle of latitude KS', so that $KS = KS'$; i.e. its latitude would remain unaltered, whilst its longitude would be increased by the $\angle SKS'$; also for a star situated as in the figure, both the right ascension and declination would be increased, but for stars in a different position, the effect would be different; a star for instance in the ecliptic near the equinoxes, would pass from one side of the equator to the other. The observed common annual increment of longitude is $50'',2$, amounting to 1^0 in $71\frac{1}{2}$ years, and making the time of revolution of the stars about Kt nearly 26 thousand years.

73. Though this hypothesis of real motions of the stars in parallel circles about the poles of the ecliptic, accounts for the phenomena, yet as the Sun, Moon and planets, which have motions independent of the general body of the stars, also exhibit those phenomena, we cannot rest satisfied with it; and we are lead to seek for a simpler explanation, in a motion of the assumed points and planes relative to which the positions of the stars are determined; such as would result from a change in the situation of the Earth's equator with respect to the plane of her orbit. Accordingly we find, by reason of the action of the Sun and Moon on the protuberant matter at the Earth's equator, that whilst the inclination of the equator to the plane of the ecliptic remains always very nearly the same, their line of intersection does not preserve its parallelism, but revolves slowly in the plane of the ecliptic in a direction *contrary* to that in which the Earth moves and rotates, viz. from east to west; and the pole of the heavens consequently describes a circle about the pole of the ecliptic in the same direction.

74. To make this clearer, let T, fig. 12, be the Earth's center, moving about S, the Sun in the direction tT, AB the intersection of its equator AQB with the plane of the orbit, passing through S the Sun, and consequently determining the first point of Aries (Art. 48). Then when the Earth's center returns to T, if the line of intersection pre-

serve its parallelism, it will again pass through S, and the first point of Aries will remain in the same place with respect to the fixed stars as before; but if it have a slow angular motion contrary to the direction of the Earth's motion, it will be brought into the position $A'B'$, and will not now pass through S, having done so when the Earth's center was at t determined by drawing St parallel to $A'B'$; consequently the first point of Aries has moved backward among the fixed stars through the $\angle \Upsilon S \Upsilon'$, and the longitudes of all stars are increased by that quantity; their latitudes depending upon the position of the ecliptic which does not alter, are of course unchanged.

Again, let Pt be the Earth's axis, and Kt perpendicular to the plane of the ecliptic; then PtK the plane of the solstitial colure being always perpendicular to $tS\Upsilon'$ will, at each equinox, pass through stars situated to the west of those it contained at the preceding, and as the $\angle PtK$ is nearly invariable, the pole of the heavens will describe about the pole of the ecliptic, a circle distant therefrom nearly $23^0.28'$.

75. Since this retrograde motion of the line of intersection of the equator and ecliptic makes the interval between its successive passages through the Sun, less than if it had remained parallel to itself, the retrograde motion accelerates the return of the equinoxes, and is therefore called the *precession* or going forward of the equinoxes.

A catalogue of stars, besides expressing their right ascensions on the year for which it is constructed, must give the yearly change or *annual precession* in right ascension, by means of which the right ascension and declination of a star may be reduced to a preceding, or brought up to a succeeding year; this is an instance of the *corrections* to be applied, of which it will be necessary forthwith to speak more at length.

Nutation of the Earth's axis.

76. We have seen that the attractive forces of the Sun and Moon on the protuberant matter at the Earth's equator produce the precession of the equinoxes and cause

the pole of the heavens to describe a circle about the pole of the ecliptic; on more minute investigation, however, it appears that the place of the pole thus ascertained does not exactly agree with its real place. The mean quantity of precession, which has been stated at $50'',23$, is that which is produced in a whole year; but it is clear that it cannot have been produced uniformly; for the efficacy of the actions of the Sun and Moon depends upon their distances from the plane of the equator which are continually varying; if both bodies continued always in that plane, there would be no precession. The Sun's action therefore must make the angular motion of the Earth's axis about the axis of the ecliptic inequable; the correction which it is from this cause necessary to apply to the mean quantity of precession, on the supposition of its being uniformly generated, to get the actual quantity at any time, is the *solar inequality of precession*. Also the Sun's action sometimes increases and sometimes diminishes the inclination of the axis of rotation to the axis of the ecliptic, without permanently altering it; the correction arising from this cause to be applied to the *mean* obliquity, to get the true obliquity at any time, is called the *solar nutation*.

77. Again, as the amount of precession produced by the Sun depends upon the inclination of the Earth's axis to the ecliptic or Sun's apparent orbit, so the amount of the lunar precession depends upon the inclination of the Earth's axis to the plane of the Moon's orbit. Now the axis of the Moon's orbit, as will be shewn, revolves about the axis of the ecliptic in about $18^{y}.7^{m}$, with a motion nearly uniform, preserving nearly a constant inclination to it of $5^{0}.9'$; and the Earth's axis revolves about the same line, at an inclination of $23^{0}.28'$, in a much longer period of 26,000 years; hence the inclination of the Earth's axis to the plane of the Moon's orbit is perpetually varying, and through a wide range; and therefore the velocity and direction of the motion of the Earth's axis produced by the Moon is also irregular; as before, the corrections which must be applied to the mean uniform precession, and mean obliquity, (as far as they are rendered necessary by the Moon's action)

are called the *lunar inequality of precession*, and the *lunar nutation*. They are very much more considerable in amount than the similar irregularities produced by the Sun.

Also, since the Moon describes a great circle about the Earth in a month, in the same manner as the Sun does in a year, there will be an inequality of precession and a nutation produced by the Moon in a month, similar to those produced by the Sun in a year; the *monthly lunar nutation* however is so small that it is seldom considered.

78. Thus it appears that the Earth's axis has a very complicated actual motion, describing, by the effect of nutation, a conical surface about its mean position, whilst the latter, by the effect of precession, describes a conical surface about the axis of the ecliptic. This deviation of the true place of the pole of the heavens from the mean place is called nutation, as it arises from an oscillation or nodding of the Earth's axis; the motion of the pole to or from any star will alter its right ascension and declination; and as the same motion affects the place of the equinox, the longitude will also be altered by it; but the latitude will remain unchanged.

Consequences of the retrograde motion of the equinoctial points.

79. Though the rate at which the equinoctial points move backwards on the ecliptic is so slow, yet the accumulated effects in the lapse of ages become very sensible; and even in a moderate number of years, Astronomers are obliged by those effects to reconstruct their Catalogues of the stars. About 2000 years ago, the 360^0 of the ecliptic were divided into twelve equal portions each of 30^0, and were named after the most remarkable constellations of the zodiac which they respectively crossed; thus the first 30^0 of the ecliptic, running across the constellation Aries, was called the sign of Aries, and the equinox was called the first point of Aries; at that time the constellations, and the divisions of the ecliptic were equally called signs of the zodiac, without causing confusion. Since that epoch, by reason of the annual increase of longitude of $50'',2$ common to all the stars, the constellations have all changed their places relative to

the divisions of the ecliptic. Though the vernal equinox is still called the first point of Aries, the circumstance which gave rise to that name exists no longer; that point of the constellation is now removed 30^0 east of the equinox, and the beginning of Pisces has taken its place. Hence it is necessary to make a distinction between the *signs* of the ecliptic, and those *groups of stars* which bear the same name; the former are spaces of 30^0 reckoned from the actual position of the vernal equinox; the latter also occupy 30^0, but are in continual motion along the ecliptic, and are at present about 30^0 in advance of the signs of the same name. The beginning of the *sign* Aries will always be in the intersection of the ecliptic and equator, the beginning of the *constellation* Aries, only after intervals of 26 thousand years; when the Sun enters the *sign* Cancer, it will always be midsummer; but he will in the course of ages enter the *constellation* Cancer at every season of the year.

Another effect of precession may be noticed in the pole star, which is at present approaching the pole, and its distance therefrom, about the year 2095, will be reduced to $26\frac{1}{2}'$; after which, in describing its circle round the pole of the ecliptic, it will continually recede from the pole of the heavens for the succeeding half period of its revolution.

80. It was stated in Art. 69, that the apogee has a progressive annual motion in the ecliptic, relative to the fixed stars, of $11'',25$; this added to $50'',23$ which is the annual retrocession of the equinoctial points, gives $61'',47$ for the quantity by which the longitude of the apogee is every year increased. The apogee at present is advanced in the ecliptic about 10^0 east of the northern solstice; about the year 1250 they coincided, and the major axis of the solar ellipse was then perpendicular to the line of equinoxes; and about 4000 years before the Christian era, i. e. about the time of the creation, the apogee was in the vernal equinox; the combined effects of their opposite motions, have since brought them into their present relative positions.

Different kinds of Years. Calendar. Julian Correction.

81. The period of the Sun's completing the circuit of the heavens is pointed out by nature for the measure of

long durations, in the same manner as his returns to the same meridian are, for determining shorter intervals; and that period may be estimated in three different ways; either as the interval between his leaving an equinox or solstice, and returning to the same equinox or solstice; or as that between his leaving any position among the fixed stars, and returning to the same position; or as that between his leaving the perigee of his orbit, or a point at a given angular distance from perigee, and returning to the perigee, or to the same angular distance from perigee; the first is called a *tropical* year, the second a *sidereal*, and the third an *anomalistic* year (for the Sun's angular distance from perigee is called his true *anomaly*); and this is the order of their lengths; for since the equinox or solstice moves backwards through $50'',2$, and the perigee advances through $11'',2$, every year along the ecliptic, the Sun describes 360^0 *minus* $50'',2$ in a tropical year, 360^0 in a sidereal, and 360^0 *plus* $11'',2$, in an anomalistic year.

82. The Sun's position with respect to the equinox, as we know, determines what season it is, and his passage between the equinoxes occasions the whole variety and succession of the seasons; hence the *tropical* year includes the entire series of natural operations and appearances, and is therefore adopted as the unit of long duration. To the same epoch of the tropical year the same state of the visible face of nature will always correspond; but the same epoch of the sidereal or anomalistic year might, in the course of ages, answer to seasons perfectly different from one another. A duration so adjusted, that it shall consist of a certain number of entire days, and that the same position of the Sun relative to the equinoxes, shall always happen nearly on the same day reckoned from its commencement, is called a *civil* year. We cannot take the exact mean length of the tropical year, which is $365^d. 5^h. 48^m. 51^s,6$, for our civil year, since it would be inconvenient to have the same day belonging to two different years. If the tropical year consisted of exactly $365\frac{1}{4}$ days, then three consecutive years of 365 days each, followed by one of 366 days would be equal to four tropical years; this was the adjustment of the calendar introduced by

Julius Cæsar; but since a tropical year is less than a Julian year of $365\frac{1}{4}$ days by $11^m.8^s$, if their commencements were separated by a known interval at any one epoch, for every following year that interval would be increased by $11^m.8^s$, which in a century would amount to $18\frac{1}{2}^h$, and in four centuries to nearly three days.

83. Accordingly, the vernal equinox which, in 325 at the council of Nice, fell on the 21st of March, in 1582 happened on the 11th of March; therefore Pope Gregory caused 10 days to be omitted, making the 15th of October immediately succeed the 4th, so that the next year the vernal equinox again fell on the 21st of March; and to prevent the recurrence of the error, it was ordained that in every succeeding cycle of four hundred years, three of the leap years should be common years; and for the purpose of being easily recollected, that the excepted leap years should be those which complete the three first centuries of the cycle. According to this adjustment 400 civil years will only differ from 400 tropical years by $2^h.16^m$, and the error will not amount to a day in four thousand years; the rule for it is this, every year whose date is divisible by four is a leap year, except that date be exactly a number of centuries, in which case, that number must be divisible by four in order that the year may be a leap year; thus 1600, 2000, 2400 are leap years; 1700, 1800, 1900, 2100 are common years. This correction was introduced into England in 1751, when it had become necessary to omit eleven days in the current year.

The civil year begins without any astronomical reason, about eleven days after the winter solstice; but the time of its commencement is of no importance; for provided its length never differ much from that of the tropical year, the same day of the year will always nearly correspond to the same position of the Sun relative to the equinox, which is the principal thing to be attended to.

Figure and dimensions of the Earth.

84. To arrive at a knowledge of the absolute distances and dimensions of the heavenly bodies in miles, we must have a standard of measure derived from our own globe; it be-

comes necessary therefore to explain how the size and figure of the Earth are determined, especially as its *spheroidal* shape has been assumed in the explanation of precession and nutation. The curvature of the Earth's surface must have been at first suggested, and is at all times indicated by the increase observed in the elevation of the pole in advancing towards the north; and as of all round figures the sphere is the simplest, the Earth was originally supposed to be spherical. That this supposition differs very little from the truth, may be shewn experimentally as follows. If an observer determine two places on the same meridian at which the difference of the meridian zenith distances of the same star is 1^0, the difference of latitude of the places is also 1^0, and the distance between them, the length of 1^0 of the meridian; now this distance has been measured in different latitudes north and south of the equator, and on different meridians, and is found to be everywhere nearly of the same length, conditions which can be satisfied only by a body differing little from a sphere; its mean length is about $69\frac{1}{22}$ miles; this multiplied by 57,295 which is the number of degrees in an arc whose length is equal to the radius, gives 3958 or nearly 4000 miles for the length of the Earth's radius.

85. However the length of a degree of the meridian determined in this way at places whose latitudes differ considerably, is found not to be exactly the same, but to *increase* from the equator to the pole, the excess of the length of a degree in any latitude over that at the equator, being nearly proportional to the square of the sine of the latitude; this, together with the analogy of some of the planets which seem to be considerably flattened at their poles, and the consideration of the form which a revolving fluid mass would assume, lead to the conclusion that the figure of the Earth is more nearly an oblate spheroid having its axis of revolution perpendicular to the equator. By the figure of the Earth is not meant that body with all its inequalities, but an imaginary figure such as would result from reducing hills and vallies to the same level, or from supposing the sea continued underneath the continents; for the heights of its mountains are very inconsiderable compared with the Earth's dimensions, and the

courses of great rivers, which are in general navigable to a great extent, prove that the curvature of the land differs but little from that of the sea.

Astronomical corrections enumerated.

86. We have seen that by the effects of precession and nutation the place of a fixed star is differently described at different times, and that we cannot consequently make use either of the *registered* right ascension for regulating the clock, or the *registered* declination for finding the latitude, without first applying the corrections for precession and nutation. This leads us to enquire if there may not also exist causes which make the *observed* place of a star differ from its true one, as for instance the action of the Earth's atmosphere on the light which proceeds from a body and makes it visible, the non-instantaneous transmission of that light combined with the spectator's moving obliquely to its direction, and the circumstance of the observer's not being at the Earth's center; we find in fact that all these do occasion errors, which are called respectively *refraction*, *aberration*, and *parallax*, and which together with precession and nutation, constitute the *five astronomical corrections* necessary to deduce from the observed or registered place of a heavenly body its true one.

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87. Though comprehended under the same name with the others, precession and nutation form a perfectly distinct class of corrections; they are not necessary to be applied to direct observations, but to registered observations, in order to obtain the accurate present values of elements, and to connect distant observations, and depend on physical Astronomy for a knowledge of their causes, and for the expressions of their laws; whereas the three others affect an observation at the time of making it, and are designed to divest it of an optical illusion; but even these are not without essential differences; for instance, by the effect of refraction and aberration we do not see the body in its real position in space, and by the effect of parallax not in its real place with respect to the fixed stars (i. e. such as it would be seen to occupy from the Earth's center, and which

is general to all observers) though in its real position in space; the two former affect all the heavenly bodies, the latter only the nearer, viz. the Moon and Planets, in a perceptible degree; again, refraction and parallax are totally independent of the motion or rest of the observer, whilst aberration depends upon his motion.

Motion of the Moon in an ellipse round the Earth.

88. Of all the heavenly bodies after the Sun, we are most interested in the Moon, and we proceed to notice her motions. It has been stated that the Moon has a proper motion from west to east among the fixed stars, and that her track coincides with the intersection of a plane through the Earth's center inclined at about 5^0 to the ecliptic, with the imaginary sphere of the heavens; this is deduced from observations of her place at different times, in the same manner as for the Sun; and by comparing her distances at those times by means of the apparent diameters, we further discover that her orbit is an ellipse with the Earth in its focus, that her radius vector describes areas nearly proportional to the times, and that the line of apsides is not stationary, but moves forward in the plane of the orbit at the mean rate of $40^0.40'.32'',2$ in every year.

89. The Moon's apparent diameter varies from $29'.22''$ to $33'.31''$ which are nearly in the ratio of 7 to 8; this consequently is the ratio of the Moon's least and greatest distances from the Earth; the mean distance is about 60 radii of the Earth, i. e. about $\frac{1}{400}$th part of the Sun's distance. The points where the plane of the Moon's orbit cuts the ecliptic are called the Moon's *nodes*, that in which she is when passing from the south to the north side of the ecliptic being called the *ascending* node, and the other the *descending* node; by observing at different times the stars which the Moon hides in crossing the ecliptic, it is found that the line of intersection of the Moon's orbit with the plane of the ecliptic, called the *line of Nodes*, has a backward angular motion on the ecliptic, amounting to $19^0.21'.23'',6$ in a year; hence, as was assumed in Art. 77, the axis of

the Moon's orbit revolves about the axis of the ecliptic in $18^{y}.7^{m}$.

90. The Moon is a round opaque body, whose diameter is about $\frac{3}{11}$ths of the Earth's, giving forth no light of its own, but reflecting the light received from the Sun. When the Sun and Moon have the same longitude, they are said to be in *conjunction;* when their longitudes differ by 180^{0}, they are said to be in *opposition;* in both cases their centers are in the same secondary to the ecliptic, but they are seen in the former case in the same, and in the latter in opposite parts of the heavens. In conjunction the Moon is between the Sun and Earth, consequently her dark side is turned towards the Earth, or it is new Moon, and both luminaries are on the meridian nearly together; in opposition the Earth is between the Sun and Moon, and the bright side of the Moon is visible, and the Moon is on the meridian at midnight.

91. The interval between two successive full Moons is called a *lunation, lunar month,* or *Moon's synodic period,* its mean length being about $29\frac{1}{2}$ days, in which the Moon traverses not only the circuit of the heavens, but also the space described by the Sun in the interval; this being allowed for, we have $27\frac{1}{3}$ days for the period between leaving and returning to the same longitude, and $13^{0}.10'.35''$ for the angle in her orbit described by the Moon in 24 hours. Hence the Moon moves towards the east 13 times faster than the Sun, and comes to the meridian later than the Sun every day by the mean duration of $48^{m}.46^{s}$. To us the appearance is the same as if the Moon described an ellipse in $27\frac{1}{3}$ days, and the Sun an ellipse 400 times as great in $365\frac{1}{4}$ days, their planes being inclined to one another at $5^{0}.9'$, and the Earth being fixed in their common focus and rotating about an axis in one of these days. But the reader will not fail to observe that the Moon's absolute motion is very different from this, arising, as it does, from her apparent motion round the Earth in an orbit which itself partakes of the Earth's progressive motion.

The Moon's Phases.

92. We proceed in the next place to describe and account for those remarkable changes in the shape of the Moon's disk called her *phases*, which recur every lunation, and to which the attention of common observers is first and chiefly directed. After having been invisible for four or five days, in the course of which we know the Moon has the same right ascension as the Sun, and comes to the meridian at the same time, she is seen just after sunset 30^0 or 40^0 east of the Sun, under the form of a slender crescent convex towards the Sun. As her easterly motion is 13 times greater than the Sun's, when first seen on each of the following evenings, her position is more and more to the east; during this time the breadth of the crescent continually increases, the convex or western boundary remaining circular, whilst the concave boundary becomes less strongly curved; until the Moon is distant 90^0 from the Sun and comes upon the meridian about six in the evening, when the eastern edge is changed into a straight line, and the phase is the *half moon*.

After this time, the Moon continuing to separate herself in an easterly direction from the Sun, the side most remote from the Sun bulges out, and the bounding curves of the illumined disk have their concavities opposed; in this state the Moon is called *gibbous;* and the breadth continually increases as well as the curvature of the remote side, till having attained her greatest distance from the Sun, viz. 180^0, and therefore crossing the meridian about midnight, she appears perfectly round or it is *full moon*.

From this time, as the distance between the luminaries diminishes, the appearances presented during its increase succeed each other in a reverse order; the breadth gradually diminishes, the Moon becomes first gibbous, then semicircular when she is 90^0 west of the Sun, and therefore comes to the meridian at 6 in the morning, then a crescent visible in the east just before sunrise, until at length having attained the same distance west of the Sun that she was east of him when first visible at the beginning of the month, she disappears, to become visible again at the accustomed period of the next month, and to exhibit the same phenomena in the same order.

At equal angular distances from the Sun, east or west, the shape of the Moon is the same; the only difference being that the western limb is illuminated in the former case, and the eastern in the latter; i. e. in all cases the illumination is on that side of the disk which is nearest the Sun.

There is no difficulty in shewing that the above appearances are exactly such as would be presented by an opaque spherical body describing an orbit round the Earth, and illuminated solely by the Sun's rays.

Moon's rotation, same face always turned towards the Earth.

93. Every one has remarked that the Moon's disk appears chequered with dark spots and lines, so distinct in their characters, that the disappearance of any of them, or an alteration in their position with respect to us, can be immediately detected. By observing these spots, it is found that the hemisphere turned towards us is always very nearly the same; therefore the Moon must rotate about an axis in the same direction as she moves in her orbit, and must complete a revolution in the same time as she completes the circuit of her orbit; just in the same manner as a person moving round any object with his face constantly turned towards it, turns himself round in the same direction, for his face is successively directed to every point of the horizon. This law is found to obtain in the case of all satellites or bodies of the same order as the Moon; their periods of rotation are equal to their times of revolution, and they consequently always turn the same face to their primaries or planets which they attend. A spectator therefore at the Moon in $27\frac{1}{3}$ of our days would have only one day and one night; if on the averted hemisphere, his long night would never be enlivened even by a Moon, but if on the hemisphere turned towards the Earth, he would enjoy the reflexion of light from the Earth during the whole absence of the Sun; and the Earth when full would present a far more splendid appearance than the Moon does to us, its surface being 13 times larger, and by its comparatively quick rotation, all the varieties of cloud, land, and water coming in rapid succession into view.

Moon's surface, mountains, and atmosphere.

94. On viewing the Moon through a powerful telescope, certain bright spots are observed, accompanied by comparatively dark shadows whose directions are always opposite to the Sun, and lengths proportional to his elevation above a plane touching the Moon at those points; just as would happen if the spots were mountains in the Moon. That they are so, is further confirmed by observing that the inner elliptic boundary is every where serrated with lines and points of light projecting beyond its general contour, and that there are even insulated peaks, situated slightly but distinctly beyond it, in the unillumined portion; exactly as in our own globe, the tops of high mountains are enlightened, when the surrounding regions are in shade. Hence we conclude that the Moon's surface is everywhere rugged, without any great plains or large spaces covered with fluid which would necessarily be smooth; this notion of the absence of evaporable fluids is strengthened by there not seeming to be any clouds, which, if they existed, ought to cast perceptible shadows. Several of the lunar mountains are supposed to exceed two miles in height, and one* or two are thought to be as much as five miles; occasionally other bright spots are seen in the dark part so far from the edge of the bright part, that they cannot be supposed to be mountains, they are therefore self-luminous, probably volcanoes.

95. The Moon is supposed to have no atmosphere, or one of extreme rarity, greater than can be produced by our best air-pumps, for the following reasons; first, the partial and faint illumination beyond the boundary of the enlightened part, which would result from an atmosphere capable of producing twilight, scarcely exists at all; secondly, the Moon in her course through the heavens is necessarily sometimes in the same direction with one or other of the fixed stars, and being much nearer, passes between it and the Earth and conceals it from us; this, which may happen to any star whose latitude does not exceed $6\frac{1}{2}^0$, is called an *occultation*, and it is found that the observed duration of an occultation is not sensibly shorter than the computed dura-

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* 22362 Paris feet, according to Beer and Mädler, *Selenographie*. Berlin, 1837.

tion, which would be the case if the Moon had an atmosphere capable of refraction; for at both apparent contacts of the star, its rays, in grazing the Moon's body, would be bent inwards, and would reach the Earth, both at the beginning of the occultation after some part of the Moon was between the Earth and star, and at the end before the whole of the Moon had passed from between them.

Proportion of Moonlight at different seasons and places.

96. The most obvious service the Moon renders to mankind is to supply light during the Sun's absence, and though she does so for only about a quarter of it, and though her stay above the horizon of any place as well as the light she affords are perpetually changing, yet the conditions of her motion and illumination are such that she always continues longer above the horizon of any place as her light increases, and the longest at those places where it is most needed. As the Moon can never deviate more than 5° from the ecliptic and to that extent only during a very small part of her course, for the sake of simplicity we will suppose her to move in the ecliptic. Between the epochs then of the vernal and autumnal equinox, the Moon when full has south declination, the greater, the nearer the season is to midsummer; at all places therefore in north latitude, when full, she is less than 12 hours above the horizon, and for the shortest time and attaining the least altitude at midsummer; similarly, for the other half of the year, when full, having north declination she will continue more than 12^h above the horizon in north latitudes, and the longer the higher the latitude, and everywhere for the longest time and attaining the greatest altitude, at midwinter; i. e. the time during which she continues above the horizon when her light is greatest, increases, during winter, as the lengths of the nights themselves increase, both at the same place for different nights, and at different places for the same night. (Arts. 20 and 21.)

97. Besides thus accommodating her stay above the horizon to the season, she will always when full, being in the point of the ecliptic opposite to the Sun, pass the meridian at

midnight, i.e. be most effective in giving light at the period of greatest darkness; and the periods of her greater light will always bring her on the meridian nearer midnight, than those of her less brilliancy. In the extreme case of the pole in the zenith, the Moon will always be above the horizon a fortnight at a time, during the whole of which, at the winter solstice, more than half the disk will be illuminated; at the summer solstice, more than half will be dark; and as the Sun approaches an equinox, the light given by the Moon during her fortnight's stay will diminish in one case, and increase in the other; it must be observed however, that as the Moon may deviate 5° from the ecliptic, her declination may be north in cases where we have supposed it south; and that consequently our conclusions are true as to their nature, but not as to their amount.

98. The motions of the Moon have become of great importance to the navigator and geographer, from the precision with which the longitude is determined by the occultations of a star, and by *lunar distances,* i.e. the Moon's distances from the Sun or a star; the Moon in fact is the only heavenly body whose motion is sufficiently quick to serve for the determination of the longitude; for the error in the result, arising from the errors of observation and of the tables, will evidently be the greater the slower the motion of the body. The times of these phenomena observed under any meridian, when compared with that computed for Greenwich in the Nautical Almanac, give the longitude of the observer within a few miles.

Proper motions of some of the stars. Planets, Kepler's Laws.

99. The stars, as has been stated, are fixed in the firmament at a prodigious remoteness, and their diurnal motion is merely an appearance caused by the Earth's rotation; several of them however have small progressive motions, especially μ Cassiopeiæ and 61 Cygni, both small stars; not so with the Planets, which though not by a careless or momentary observation distinguishable from the fixed stars, are soon discovered to be incomparably nearer to us, to have proper motions like the Sun and Moon, to be round opaque bodies shining with

light borrowed from the Sun and rotating about an axis, and describing elliptical orbits in different planes with the Sun in their common focus, but all inclined at a small angle to the ecliptic. There are seven principal planets, called in the order of their distances from the Sun, Mercury, Venus, the Earth, Mars, Jupiter, Saturn, and Uranus. Mercury and Venus, since their orbits lie within that of the Earth are called *inferior* planets, and present phenomena different from the others; they are seen in *inferior* conjunction when between the Earth and Sun, and in *superior* conjunction when the Sun is between them and the Earth, but never in opposition. There are also four smaller planets, Vesta, Juno, Ceres, and Pallas, which were discovered in the first seven years of the present century; their orbits lie between those of Mars and Jupiter.

100. The motions of all the planets in their orbits are from west to east, or according to the order of the signs, and are all subject to Kepler's three laws, which we have already seen to be obeyed by the Moon and Earth, and which by similar observations are found to hold in the present instances; viz.

(1) The radii vectores describe areas in one plane proportional to the time. R
(2) The orbits are ellipses having the Sun in one of their foci.
(3) The squares of the periodic times are proportional to the cubes of the mean distances from the Sun.

The first shews that the force which acts on the bodies tends always to the center of the Sun; the second, that the force varies inversely as the square of the distance; the third, that the absolute force is the same for all planets, and consequently that their masses may be neglected compared with that of the Sun; another statement of this third law is that the area swept out in a given time by the radius vector of any body is proportional to the square root of the latus rectum of the orbit, under which form the law is applicable to the parabolic orbits of comets also. The laws were established by Kepler from his own observations and those of Tycho Brahe;

and the consequences demonstrated by Newton; their discovery is an era of vast importance in the science of Astronomy, and their truth the basis of the universal principle of gravitation, viz. that all the particles of matter mutually attract each other as their masses directly, and as the squares of their distances inversely.

101. The true motion of the planets, by reason of their mutual attractions, are very complicated; they do not in reality move in any known or symmetrical curve, but the overpowering force of the Sun causes ellipses, nearly approaching to circles, to be the closest approximation to their paths. The planes of all the orbits, except those of the recently discovered planets, are inclined at small angles to the ecliptic, and cut it in straight lines passing through the center of the Sun. The lines of Apsides of all the orbits have direct angular motions in the planes of the respective orbits, (with the sole exception of the line of apsides of Venus, which revolves in a retrograde direction) and the lines of Nodes of all the orbits have retrograde angular motions upon the ecliptic; agreeing in these respects with the orbits of the Earth and Moon. Moreover the inclination and eccentricity of every orbit are in a state of perpetual but slow change; the eccentricity of the Earth's orbit at present diminishes at the rate of nearly a radius of the Earth in a century, and the inclinations of all the planetary orbits are decreasing, but the mean distances and mean motions remain permanently unaffected by secular changes. All the variations of the solar system are periodical, that is, after increasing to a certain amount they diminish by the same slow changes, and for ever oscillate about a mean value; in this consists the stability of our system.

102. The angles of elongation of the inferior planets from the Sun can never exceed a certain limit, which is about 47^0 for Venus, and $28^0,8$ for Mercury; sometimes, when they are to the east of the Sun, they are seen in the west just after sunset, when they are called *evening* stars; at other times, when they occupy the opposite points of their orbits, or are to the west of the Sun, they are seen in the east a little before sunrise, when they are called *morning* stars; but they are never visible at other times, or in other positions to the naked eye, (Venus excepted, which

is visible in daylight for a short time every eighteen months); for being so near the Sun, their orbits are carried along with the Sun, in the same manner as the Earth carries the Moon's orbit along with her, in her annual course. For the superior planets, the angles of elongation admit of all magnitudes; consequently they are seen at all hours of the night, and at various altitudes above the horizon; when they are in opposition they pass the meridian at midnight. The inferior planets at their inferior conjunctions are occasionally seen to pass over the Sun's disk like black spots, proving thereby that they are opaque bodies shining by reflected light; these passages, which are called *transits*, can never happen for the superior planets, because the latter can never pass between the Sun and Earth. The interval between two consecutive oppositions or conjunctions of a superior planet, or between two conjunctions of the same kind of an inferior planet, is called the *synodic* period.

103. The combination of the Earth's motion in her orbit with a planet's motion, causes apparent irregularities in the latter, sometimes making the planet seem to move from east to west or to be retrograde, and sometimes to seem stationary. When an inferior planet is near inferior conjunction, its apparent motion from west to east gradually diminishes till it becomes stationary; it then moves backwards, that is, from east to west till conjunction, and continues to move with a decreasing velocity in the same direction for an equal time after conjunction, when it again becomes stationary; it then starts again in a direct course, and goes on in that direction through superior conjunction for nearly half a synodic revolution, when it becomes stationary and then retrogrades through inferior conjunction; that is, an inferior planet appears always to be moving backwards when in the conjunction nearest the Earth, and forwards when in that furthest from the Earth. Similarly a superior planet always appears to be moving backwards at opposition, and forwards at conjunction.

104. The rotation of the planets is from west to east like their revolutions; Mercury, Venus, the Earth, and Mars accomplish their rotations in about 24 hours, whilst Jupiter and Saturn perform theirs in $\frac{4}{10}$ths of a day.

There appears to be a relation between the distances of the planets from the Sun first noticed by Bode, who remarked that a planet was wanting where the new planets have since been discovered; the Earth's distance from the Sun being 10, the distances of the planets are nearly as follows:

Mercury	4 =	4.
Venus	$4 + 3.1$ =	7.
Earth	$4 + 3.2$ =	10.
Mars	$4 + 3.2^{2}$ =	16.
New planets	$4 + 3.2^{3}$ =	28.
Jupiter	$4 + 3.2^{4}$ =	52.
Saturn	$4 + 3.2^{5}$ =	100.
Uranus	$4 + 3.2^{6}$ =	196.

105. We shall now separately notice the chief peculiarities of the Planets taken in order.

The motions of Mercury are less disturbed than those of any other body on account of his proximity to the Sun; and his orbit is by far the most eccentric of those of the principal planets. His transits recur at intervals of 7, 13, 33, &c. years.

The great service rendered by Venus is to determine the *Sun's parallax*, i. e. the angle which the Earth's radius subtends at the Sun's center, that important element on which the knowledge of the dimensions of the whole solar system depends; and which being less than nine seconds requires methods of far greater nicety and exactness than those which succeed tolerably for the Moon whose parallax is nearly a degree, and can only be satisfactorily determined on the occurrence of rare phenomena such as the transit of a planet over the Sun's disk. As the Sun is not so remote, but that the Earth's radius has a sensible magnitude when viewed from his center, the chord described by Venus at one of her transits and consequently the time, varies with the position of the spectator on the Earth's surface. Thus, let V (fig. 23.) be Venus, W, T, two places on the Earth's surface, AB the Sun's disk; then the true line of transit, seen from E the center, is AB, and at T

and *W* the transit seems to be along the chords *ab*, *a′b′*, of which *ab* being further from the center will be described in a shorter time than *a′b′*; the difference of duration, being entirely the effect of parallax, furnishes the means of computing the difference of parallax; also the ratio of the distances, and therefore of the parallaxes, of the Sun and Venus is known from the periodic times; hence the Sun's parallax is found.

The returns of the transits of Venus depend on the ratio of the periodic time of Venus to that of the Earth being nearly as 8 : 13, or as 235 : 382. Hence a transit may happen at the same node after an interval of eight years, but if it does not happen then, it cannot take place again at the same node for 235 years; the next transits will happen in 1874 and 1882 in the month of December, at the ascending node.

The orbit of Mars immediately incloses that of the Earth; his disk is occasionally gibbous. Spots near his pole that augment or diminish according as they are withdrawn from or presented to the Sun are supposed to be masses of ice.

Vesta, Juno, Ceres, and Pallas are all nearly at the same distance from the Sun, and consequently their periodic times are nearly equal; also the eccentricities of the orbits of Juno and Vesta, and the position of their nodes, are nearly the same; for these and other reasons they have been supposed to be fragments of a single planet that once revolved about the Sun at their common distance. Though they have no sensible effect in troubling the other bodies, their own motions are made very irregular by the proximity and vast magnitude of Jupiter and Saturn. They are invisible to the naked eye, and so minute that their apparent diameters have not yet been measured; Juno the largest is supposed to have a real diameter of about 200 miles.

Jupiter the largest planet in the system is attended by four satellites, discovered by Gallileo in 1610, and may be regarded as an epitome of the solar system. His rotation is so rapid that a point in his equator moves 26 times faster than a point in the Earth's equator; his compression is consequently very large. The eclipses of the satellites afford the easiest if not the most accurate method of finding terrestrial longitudes, and one which can at all times be used, on account of the frequent occurrence of an eclipse.

Saturn seen through a telescope offers the most striking spectacle of all the planets. He is surrounded by a ring divided into two parts by a concentric dark band; it has lately been discovered that the rings are not concentric with the planet; they exhibit a variety of appearances depending upon the position of the planet with respect to the Sun and Earth, but are usually of an elliptical shape. Besides the rings, Saturn is attended by seven satellites.

Uranus was discovered by Herschel in 1781; it is accompanied by six satellites only visible with the best telescopes; its time of rotation has not been determined.

Comets, their peculiarities.

106. Comets are luminous bodies occasionally appearing in the heavens but never at a great distance from the Sun. They are distinguished from the planets in the following respects: (1) they are less bright, having somewhat of a nebulous appearance which is supposed to be owing to vapours raised by the solar heat at their surfaces in their perihelion passages: (2) they never continue visible for more than a few months at the most: (3) they exhibit no phases even when viewed with high magnifying powers, although it is probable they receive their light from the Sun: (4) the directions of the motions about the Sun is in some instances retrograde: (5) their paths have every possible inclination to the plane of the ecliptic: (6) they often exhibit tails of light which chiefly makes them objects of general attention; these tails are frequently of great length but of such small density that stars are visible through them, and are generally situated in the planes of the orbits: they follow the comets in their descent towards the Sun, but precede them with a slight curvature in their return. Their masses are so minute that they have no sensible diameters, the nucleus being principally formed of the denser strata of the nebulous matter.

CHAPTER II.

ON ASTRONOMICAL INSTRUMENTS, AND THE FIGURE OF THE EARTH.

Description of the Transit Instrument.

107. THE Transit, as has been said, is a meridianal instrument for observing, with the aid of a clock, the time when a heavenly body crosses the meridian of the place of observation. It consists of a telescope *AB*, fig. 4, fixed to two arms, the extremities of which are formed into two perfectly equal cylindrical pivots having their axes in the same straight line, which is called the *axis of the instrument*. Parallel and perpendicular to this axis, at the principal focus of the object glass, are usually fixed one horizontal wire or thread, and seven vertical wires, so as to be visible through the eye piece with the image of the celestial body in the field of view; an apparatus of this kind is indispensable to the precision of an observation; for as the field of view is not a mere point, there does not exist a single position of the telescope in which only the star can be seen. The pivots rest in angular cavities formed in pieces of metal which are attached to immoveable pillars *E* and *W*, of equal height placed as nearly as possible east and west, and are capable of small motions by screws, one in a vertical and the other in a horizontal direction. When the transit is in perfect adjustment, the line joining the center of the object glass with the point in which the horizontal wire is intersected by the imaginary wire which is the mean of the seven vertical wires in position, which line is called the *line of collimation*, traces out the meridian plane of the place of observation.

108. The errors of adjustment of the Transit consist in the axis not being exactly perpendicular to the plane of the

meridian, and in the line of collimation not being exactly perpendicular to the axis.

The inclination of the axis of the Transit to the horizon is the *error of level;* it is remedied by elevating or depressing one of the ends of the axis, and so making it horizontal; the horizontality being determined by the bubble of a spirit level L (whose tube is suspended parallel to the axis of the instrument by equal parallel vertical arms from the pivots of the axis) remaining stationary, when the position of the level is reversed.

The complement of the inclination of the line of collimation to the axis of the instrument is the *error of collimation;* it is remedied by shifting horizontally the frame in which the wires are fixed, till a small well-defined object seen on the central wire (supposing that to coincide with the mean wire) is still seen there after the telescope has been lifted off its angular bearings, reversed, and again directed to the object; this can evidently never happen till the line of collimation is strictly perpendicular to the horizontal axis; for if it deviated towards the left, after reversion, it would deviate just as much towards the right; and the object which before reversion was bisected by the central wire, would after reversion appear to the left of it.

The complement of the inclination of the axis to the meridian is the *error of deviation* or the meridian error. It is removed by altering in azimuth the position of one of the supports of the axis; for by this means the line of collimation which, after the errors of level and collimation have been corrected, describes a vertical plane nearly coinciding with the meridian, is made actually to coincide with the meridian plane; that coincidence being determined by the circumstance of the plane described by the line of collimation exactly bisecting the diurnal circle of a circumpolar star, so that the observed interval between the superior and inferior transit is precisely half the interval between two successive transits of the same kind.

In order to ascertain the permanence of the adjustments, it is convenient to have a well defined distant object in the meridian plane thus determined, as a meridian mark; this ought always to be bisected by the mean or central wire, when the instrument is in a state of perfect adjustment.

109. Certain stars, in number sixty, from their having been observed so carefully and perpetually that their positions may be considered known with the greatest accuracy to which we can attain, are denominated *known stars*. These stars therefore may be employed in determining the errors of adjustment of a Transit, or the error of the clock, assuming the difference between their observed and registered time of transit to be entirely due to these errors. For these purposes their right ascension and north polar distance are registered in the Nautical Almanac.

Method of observing with the Transit.

110. To one end of the horizontal axis is fixed an index *n*, the extremity of which, as the telescope revolves, sweeps the graduated face of a small circle attached to the pillar; the use of this is to adjust roughly the telescope to the zenith or polar distance of a star the transit of which is to be observed*. By this means such an elevation is given to the instrument that the star enters the field of view nearly upon the horizontal wire, and the instant at which it is bisected by the mean wire is the time of transit.

In observing at night, the wires are illuminated by the light of a lamp admitted into the telescope by an aperture capable of being enlarged or contracted, and reflected down the tube by a plane mirror within it, having in its center an oval aperture through which the body observed is seen. The telescope being pointed by the setting circle for the approximately known position of the body to be observed, a second is taken from the clock face before the body enters the field, and the beats are counted by the ear during its passage. The time is noted at which the body crosses the intersection of each vertical wire with the horizontal wire, parts of a second being set down by estimation; and the clock face is finally read for assurance that a second has not been missed in counting. One seventh of the sum of these times of transit is the time of transit over the mean wire. If the clock shew sidereal time, the time of transit over the mean wire, corrected if necessary for the errors of the instrument, is the right ascension, in time, of the body observed.

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* This, in large instruments is effected by means of a small graduated circle fixed at the eye-piece perpendicular to the axis of the Transit, which circle has a spirit-level moveable about its center parallel to its plane. By pointing one end of the level to the graduation equal to the nearly known meridian altitude of a heavenly body, and then turning the telescope about its axis until the bubble shews the level to be horizontal, the telescope is directed to the body.

Description of the Mural Circle.

111. The mural circle is an instrument for determining the meridian zenith distance of a heavenly body. It consists of a circle having perpendicular to its plane a concentric axis which works in a horizontal socket built into a solid wall standing north and south, so that the circle revolves permanently in the plane of the meridian parallel to the eastern face of the wall. Firmly fixed to the circle is a telescope, having in the principal focus of its object glass one horizontal and five vertical wires with a horizontal wire moveable vertically across the field by a screw with a graduated head, so that these and the image of a heavenly body to which the telescope is pointed are visible together through the eye piece. The line joining the center of the object glass with the point of intersection of the fixed horizontal wire and the imaginary wire which is the mean of the five vertical wires in position, is called the line of collimation, and when the instrument is in adjustment traces out the meridian plane of the place of observation. The limb of the instrument is divided from 0^0 to 360^0 into equal spaces of $5'$ on its outer rim which is perpendicular to the wall; degrees and minutes within five are given by a pointer fixed to the wall; and the additional minutes and seconds are read by six micrometer microscopes affixed to the wall at nearly equal distances along the limb in pairs opposite to one another, each pair having their axes in the direction of a diameter of the circle; and the microscopes are so adjusted that the image of the divided limb and the spider lines placed in the focus of the eye piece of each, are distinctly visible together; and then by a contrivance called a

micrometer, the precise point between two divisions of the limb which is bisected by the intersection of the spider lines can be estimated with extreme accuracy. The axis upon which the circle is centered is a solid truncated cone of considerable length going quite through the wall, and in contact with the socket only at the ends thereof, where it works in adjustable collars of which the eastern admits of an azimuthal, and the western of both a vertical and horizontal motion. The circle can be clamped and then moved slowly with tangent screws.

112. The mural circle is liable to the same three errors of adjustment as the transit, which are remedied by shifting the wire frame, and altering the adjustable collars in which the axis works. Thus by elevating or depressing the western bearing piece, we can *level the axis;* the test being that a plumb line suspended in front of the circle exactly over the center, and viewed through a microscope carried by the circle, when the microscope is in its upper position, must occupy precisely the same place relative to the wires in the lower position of the microscope, when the circle has been turned half round. Also by giving the western end of the axis the requisite horizontal motion, the line of collimation (supposing it perpendicular to the axis of the instrument) may be made to move in the plane of the meridian. As the circle cannot be reversed, the verification of the position of the line of collimation cannot be effected as in the transit instrument; where the latter instrument is at hand, the easiest way to perform both the adjustment to the meridian, and that for collimation, is to compare the two instruments; that is, to adjust the mural circle so that a star which is on the middle wire of the transit, may also be on the middle wire of the telescope of the mural circle.

The mural circle may at first appear, with the aid of a clock, to be capable of performing the duty of a transit, as well as its own; but the latter instrument is requisite in an Observatory, because from its lightness it is a more delicate instrument for its proper function, and because being capable of reversion it can be made to exhibit its own error of collimation, which the circle cannot do.

Method of observing with the Circle.

113. These adjustments being performed, the instrument will suffice to determine the *differences* of the meridian zenith distances of the heavenly bodies, but not immediately their absolute zenith distances. The latter is done by taking observations of the reflected image of a star from a trough of mercury, in addition to observations by direct vision in the usual way. Let abb', fig. 5, be the rim which carries the divisions, A, B, the horizontal reading microscopes, O the zero point of the graduation. Elevate the telescope so that when a star S is in the plane of the instrument, its image may be formed in the intersection of the horizontal wire and the central vertical wire; and let Ob, Oba, be the readings-off at the microscopes B, A, that is, the number of degrees, minutes, and seconds, in those arcs. Now depress the telescope, so that when the star next comes into the plane of the instrument, its image, reflected from the surface of the mercury in the trough N, may again be seen in the intersection of the horizontal wire and the central vertical wire; and let Ob', $Ob'a'$, be the readings-off. Let ab, $a'b'$, intersect in C; draw CD, CD', respectively parallel to the lines joining the center of the object glass and the intersection of the wires, when the star is observed by direct vision, and by reflection; from the point where CD' meets the surface of the mercury, draw NZ vertical, meeting CD in Z, and let NS' be parallel to CS; then because by turning the circle round its axis, the $\angle b'CD$ is made to coincide with $\angle bCD'$, $2\angle DCD' = 2\angle bCb'$ = angle subtended at the center by the sum of the arcs aa', bb'.

Hence twice zenith distance of $S = 2ZNS' = 2CNZ$

$= 180^0 - DCD' = 180^0 - \frac{1}{2}$ (angle subtended by $aa' + bb'$),

and $\therefore$ twice its altitude $= \frac{1}{2}$ (angle subtended by $aa' + bb'$)

= the *mean* of the difference of the first and second readings at each of the microscopes. down to this +

The practical method of observing with a large instrument is as follows.

114. To find the meridian zenith distance of a heavenly body by a single observation with the Mural Circle.

Set the telescope by the pointer for the approximately known meridian zenith distance of the body, and clamp it. When the body enters the field, turn the circle by the tangent screws so that it may be on the horizontal wire at its passage across the central vertical wire; then one sixth of the sum of the microscope readings is the additional minutes and seconds to be added to the indication of the pointer to give the complete reading of the circle. If the *zenith point*, or reading of the circle when the line of collimation is vertical and object glass upwards, be known, the difference between this and the former reading of the circle is the meridian zenith distance of the body.

115. To find the meridian zenith distance of a heavenly body by a double observation with the Mural Circle.

Set the telescope by the pointer for the approximately known zenith distance of the body, so that the image of the body formed by reflection in a trough of mercury may be in the field at its passage over the meridian; clamp the circle, and then read the pointer and microscopes. When the body's image is near the central wire, bisect it by the micrometer wire, unclamp the circle, and turn it until the body, seen directly, is bisected by the horizontal wire; clamp the circle and again read the pointer and microscopes. The reading of the micrometer added with its proper sign to the former reading of the circle, is the reading which would have been obtained if the body's image had been bisected by the horizontal wire; the semi-difference of this and the reading in the direct observation is the meridian altitude of the body.

116. The zenith point is the reading of the mural circle when the line of collimation of the telescope is vertical and object glass upwards. The mean of two observations of the same star by direct vision and reflection gives a reading when the telescope is horizontal; this increased or diminished by 90°, as the case may be, gives the zenith point. The practical method is this. Stars observed directly and by reflection, are divided into three groups, one comprehending stars near the zenith, the others comprehending stars far from the zenith, North and South respectively; the mean of the zenith points given by each group is considered as one result, and the mean of the three results is taken for the zenith point.

117. By using reading microscopes in pairs, any error which would arise from the center of rotation of the circle not being exactly the center of the graduation is eliminated.

Let C' be the center of the graduated rim; and C the center of motion, A, B, a pair of microscopes, $DD_{,}$, $D'D''$ positions of the telescope when viewing a body directly and by reflection, as before; then

$$\begin{aligned} 2\angle DCD' &= DCD' + D_{,}CD'' \\ &= DD_{,}D' + D''D'D_{,} + D_{,}D'D'' + DD_{,}D' \\ &= 2\,DD_{,}D' + 2\,D''D'D_{,} \\ &= DC'D' + D''C'D_{,}, \end{aligned}$$

so that the semi-sum of the readings of the two microscopes gives the angular distance between the two positions of the telescope, whatever be the position of C.

Three pairs of microscopes are employed in order to use the mean of three determinations; they are disposed at nearly equal intervals to correct the errors of graduation as well as the effect of unequal expansion of the circle.

118. When a single view of any heavenly body is all that is required, it is sufficient that the telescope should be capable of motions in altitude and azimuth; but when the body is to be kept in the field of view for some time to afford the opportunity of examining it more minutely, as in the case of micrometrical measurement of the diameter of a planet or of the distance between two contiguous stars, it becomes necessary that the telescope should have such a motion as will enable the observer to follow the body through its circle of declination without the trouble of making repeated adjustments for new altitudes. We proceed to give some account of an instrument of this description, which is essential for observations on the planets and comets.

Description of the Equatoreal Instrument.

119. The Equatoreal Instrument consists of a telescope Tt (fig. 24.) permanently fixed to the graduated limb of the

circle *MD*, and having its line of collimation parallel to the plane of the circle. At the principal focus of the object glass are five wires parallel to the plane of the circle and one perpendicular to it, and the intersection of the latter with the central one of the former set, is so adjusted that the straight line joining it and the center of the object glass, called the line of collimation, is parallel to the plane of the circle. There is also at the principal focus a micrometer of a kind depending on the nature of the measures which the instrument is designed to make. The circle above-mentioned, called the declination circle, is moveable about an axis perpendicular to its plane, whose cylindrical pivots work in adjustable collars, each of which is supported between two cylindrical pillars. These pairs of pillars are inserted at both extremities into parallel barred pieces of metal, and so form a strong frame work for enclosing and sustaining the declination circle and its attached telescope. The frame work has springing from its two parallel ends cylindrical or conical pivots whose axes are in the same straight line, called the polar axis of the instrument; the lower of these pivots works in a socket or in a hemispherical cup let into the stone pier *S*, and the upper in an adjustable collar attached by a short cast iron neck to the immoveable support *N*; the iron neck is perforated in such a manner that the stars may be seen through the apertures, when the telescope is directed towards it. Also upon the lower pivot immediately below the base of the frame work is firmly centered a graduated circle *EQ* called the hour circle, whose plane is perpendicular to the polar axis. The frame work supports two arms bearing microscopes *A*, *B*, for reading the divisions of the declination circle, which are upon its face; there are also two opposite microscopes, of which *C* is one, for reading the divisions upon the rim of the hour circle.

When the instrument is in a state of perfect adjustment, the polar axis is exactly parallel to that of the Earth, and perpendicular to the plane of the hour circle, and to the axis of the declination circle; also the line of collimation of the telescope is perpendicular to the axis of the declination circle. In this state the declination circle can be turned about the polar axis so as to coincide with the plane of any meridian, and then about its own axis till the line of collimation of the telescope makes any angle with the Earth's axis; thus any point whose right ascension and declination are known can be brought into the field of view, and if both circles be then clamped, by turning the screw which gives a slow motion to the hour circle, any star or planet may be pursued.

If the polar axis, instead of being inclined so as to be parallel to the Earth's axis, be vertical, the Equatoreal then becomes an Altitude and Azimuth Instrument.

Uses of the Equatoreal.

120. If an Equatoreal be in perfect adjustment, or if its errors of adjustment be permanent and known, then if a heavenly body be seen in the direction of the line of collimation, the readings of the hour circle and declination circle will at once give the body's right ascension and north polar distance. But this use of the instrument is little to be relied upon, because its errors of adjustment will seldom be the same in different positions, from the mechanical action of its parts. Its legitimate use is as a micrometrical instrument, for measuring double stars, or the diameters of planets; or for determining the position of an unknown body by micrometrical measurement of the difference of north polar distance of it and another known body, and the difference between their times of transit given by a clock. All these being differential observations and made in a fixed position of the instrument, are independent of its errors of adjustment. The Equatoreal is also the instrument for observing eclipses and occultations with the aid of a clock.

Description of the Micrometer Microscope.

121. The Micrometer Microscope is a contrivance for estimating the position of a point between two divisions of the graduated limb of an instrument. The microscope is placed with its axis perpendicular to the limb *AB*, so that the limb is distinctly visible through it; and the field of view is the distance between two successive graduations. Suppose for example the limb graduated at intervals of 5′. The appearance presented on looking into the microscope is that given in fig. 7. Plate II. At the principal focus of the eye-glass through the center of the field is fixed a wire parallel to the graduations of the limb defining the point of the limb whose position is to be

estimated, and above are five fixed teeth, so that each corresponds to 1$'$ of the limb. At the focus of the eye-glass there are also two spider lines intersecting one another and moveable across the field by a micrometer screw, by one revolution of which they are moved over 1$'$ of the limb, so that if the screw head be divided into 60 equal parts the reading off of it gives seconds. Suppose *ab* to be the graduation of the limb in the field of view, which suppose for example 45^0. 35$'$. The intersection of the spider lines is made to fall upon the fixed wire, in which case the screw-head shews zero, and is then moved by the screw until it falls upon the image of *ab*. Two of the teeth having been passed over by the spider lines in this motion, indicates 2$'$; and if the reading off of the screw head be 34, then, supposing the graduation to extend from right to left, the reading off of the point to be ascertained is 45^0. 37$'$. 34$''$.

Description of the Circular Micrometer.

122. The circular micrometer is a flat steel ring fastened in the center of a disc of glass, which ring is truly circular and made thin at both the exterior and interior edge. This ring, when the disc that holds it is in the eye-piece of a telescope, appears in the field of view as if suspended in the air. When it is applied to a telescope, and the angle which its interior diameter subtends at the center of the object-glass is determined, it may be used to compare the place of a planet or comet with that of a known star situated nearly in the same parallel. For, the telescope being directed to the body, let the body describe a chord of the inner circle from P to P' (fig. 8.), and let t, t', be the times of ingress and egress; again, the telescope remaining undisturbed, let T, T', be the times of ingress and egress of the star at S and S'; then the difference of the times of passage over the meridian NS which bisects both chords perpendicularly, will be $\frac{1}{2}(T'-T)-\frac{1}{2}(t'-t)$, which will be the difference of the right ascensions in sidereal time. Again, to find the difference of declination, call δ' the estimated declination of the planet, δ the declination of the star; then for the semi-chords we have

$$PH=\tfrac{1}{2}(t'-t)\,15\cos\delta',$$

$$SK=\tfrac{1}{2}(T'-T)\,15\cos\delta,$$

from which, since the angle corresponding to the radius CP is known, CH, CK, may be found; and thence the difference of declination HK.

On the figure of the Earth.

R 123. The shape of our own globe necessarily claimed a brief notice in the Introduction; and some of the obvious considerations from which its roundness may be inferred, as also the reasons for supposing it nearly an oblate spheroid were pointed out. It remains to describe the modes of ascertaining its true figure and dimensions, the chief of which consist in measuring arcs of meridians and parallels in different latitudes. Operations of that nature are of considerable difficulty and have been carried on by the ablest of modern astronomers, furnished with the best instruments; and to their works* the reader is referred for an account of the principal measures that have been made. We shall here shew, how the measures of the meridians and parallels when made, may be used in ascertaining the figure of the Earth. Either of these may be employed, but the former have generally been preferred because the differences of latitude which they require, can be observed with far greater accuracy than the differences of longitude required in the latter.

124. *The difference of the meridian zenith distances of the same star is equal to the difference of latitude of the places of observation.*

Let pe (fig. 4.) be a terrestrial meridian; Ce, Cp, the equatoreal and polar radii, ZAG, $Z'A'G'$, normals at the points A, A', meeting in O; therefore, AGe (l) and $A'G'e$ (l') are the latitudes of A and A'. Let ZAS (z) and $Z'A'S'$ (z') be the zenith distances of the same star; therefore, on account of the great distance of the fixed stars, AS is parallel to $A'S'$;

$$\therefore\ z' = Z'A'S' = Z'HS = z + A'OA = z + (l' - l);$$

$$\text{or } z' - z = l' - l \text{ or } A'OA.$$

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* *Philosophical Transactions*, for 1787, 1790, 1803, 1818. Delambre *Base du Système Metrique*. Bessel *Astron. Nachr.* No. 333. The Article in the *Encycl. Metrop.*

Hence, by observing at what places, on the same meridian, the difference of the meridian zenith distances of a star is one degree, and by measuring the arc between them, the length of an arc of one degree at that part of the meridian is determined. If the terrestrial meridian were circular, *O* would coincide with *C*, and the angle of one degree would everywhere be subtended by an arc of the same length; but the length of a degree is found to increase as we advance from the equator towards the pole, and since it may be considered to coincide with the arc of one degree of the circle of curvature at its middle point, it follows that the radius of that circle must increase, or the curvature of the meridian diminish, from the equator to the pole.

125. *To find the radius of curvature at any point of the terrestrial meridian, supposing the Earth to be an oblate spheroid.*

Let pe (fig. 5.) be the elliptic meridian; Ce (a) the equatoreal, Cp (b) the polar radius; PG (n) a normal at any point P, and r the radius of curvature at that point, $\angle PGN = l$, the eccentricity $= e$, and the difference of the semi-axes $a - b = c$. Then (Conic Sections, Art. 136.)

$$n = \frac{b^2}{a\sqrt{1 - e^2 \sin^2 l}};$$

$$\therefore\ r = \frac{(\text{normal})^3}{(\frac{1}{2}\text{ latus rectum})^2} = \frac{b^2}{a\,(1 - e^2 \sin^2 l)^{\frac{3}{2}}}$$

$$= a\,(1 - e^2)\,(1 + \frac{3}{2} e^2 \sin^2 l) = a\,(1 - e^2 + \frac{3}{2} e^2 \sin^2 l),$$

neglecting e^4, &c.

$$\text{But } e^2 = \frac{a^2 - b^2}{a^2} = \frac{a^2 - (a - c)^2}{a^2} = \frac{2c}{a} \text{ nearly;}$$

$$\therefore\ r = a - 2c + 3c \sin^2 l.$$

126. Also $CN = \dfrac{a^2}{b^2} GN = \dfrac{a^2}{b^2} n \cos l = \dfrac{a \cos l}{\sqrt{1 - e^2 \sin^2 l}}$

$$= a \cos l\,(1 + \tfrac{1}{2} e^2 \sin^2 l) = \cos l\,(a + c \sin^2 l),$$

the radius of the parallel passing through P.

$$\text{And } CP^2 = a^2 \cdot \frac{\overline{1 - e^2}|^2 \cdot \sin^2 l + \cos^2 l}{1 - e^2 \sin^2 l}$$

$$= a^2 \left\{(1 - 2e^2 \sin^2 l) \cdot (1 + e^2 \sin^2 l)\right\} \text{ nearly}$$

$$= a^2 (1 - e^2 \sin^2 l);$$

$$\therefore\ CP = a \cdot \left\{1 - \tfrac{1}{2} e^2 \sin^2 l\right\} = a - c \sin^2 l,$$

the distance of a point on the surface from the Earth's center, in terms of the latitude of the point.

127. *Given the latitude of a place on the Earth's surface considered as an oblate spheroid, to find the latitude measured from the center.*

Let $l = PGe$, (fig. 5.) be the latitude of P, $l' = PCe$ the latitude from the center of the Earth C, Ce the equatoreal radius $= a$, and Cp the polar radius $= b$.

$$\text{Then } \frac{\tan l'}{\tan l} = \frac{GN}{CN} = \frac{b^2}{a^2};$$

$$\therefore\ \tan l' = \frac{b^2}{a^2} \cdot \tan l = n \cdot \tan l, \text{ (making } \frac{b^2}{a^2} = n\text{)};$$

Hence, replacing the tangents by their exponential values (Trig. Art. 144.) we find by a well-known process,

$$l' = l - \frac{1-n}{1+n} \cdot \sin 2l + \tfrac{1}{2} \cdot \left(\frac{1-n}{1+n}\right)^2 \cdot \sin 4l - \&c.;$$

$$\text{or } l' = l - \frac{a^2 - b^2}{a^2 + b^2} \cdot \sin 2l + \tfrac{1}{2} \cdot \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 \cdot \sin 4l - \&c.$$

128. Since $\tan l = \frac{1}{n} \cdot \tan l'$, it may be shewn, in the same manner, that

$$l = l' - m' \cdot \sin 2l' + \tfrac{1}{2} m'^2 \sin 4l - \&c.,$$

$$\text{where } m' = \frac{1 - \frac{1}{n}}{1 + \frac{1}{n}} = -\frac{1-n}{1+n};$$

$$\therefore\ l = l' + \frac{1-n}{1+n} \,.\, \sin 2l' + \tfrac{1}{2} \,.\, \left(\frac{1-n}{1+n}\right)^2 .\, \sin 4l' + \&c.$$

$$\text{If } n = \cos 2\epsilon,\ \frac{1-n}{1+n} = \tan^2 \epsilon\,;$$

$$\therefore\ l' = l - \tan^2 \epsilon \,.\, \sin 2\,l + \tfrac{1}{2} \,.\, \tan^4 \epsilon \,.\, \sin 4\,l - \&c.$$

$$l = l + \tan^2 \epsilon \,.\, \sin 2\,l' + \tfrac{1}{2} \,.\, \tan^4 \epsilon \,.\, \sin 4\,l' + \&c.$$

129. The $\angle PGe$ is the *true* latitude or observed elevation of the pole, the $\angle PCe$ is called the *corrected* latitude; the difference between the true and corrected latitude is called the *angle of the vertical;* let it $= \phi$; $\therefore$ $\tan \phi = \tan (l - l')$

$$= \frac{\tan l \left(1 - \dfrac{b^2}{a^2}\right)}{1 + \dfrac{b^2}{a^2} \tan^2 l} = \frac{e^2 \sin l \cos l}{\cos^2 l + (1 - e^2) \sin^2 l} = \frac{e^2 \sin l \cos l}{1 - e^2 \sin^2 l};$$

therefore, considering e very small,

$$\phi = e^2 \sin l \cos l = \frac{e^2}{2} \sin 2\,l = \frac{c}{a} \sin 2\,l.$$

130. As a knowledge of the true figure of the Earth is of great importance, the measurement of large arcs of the meridian has been executed with great accuracy in most European countries, and in British India. The whole distance between the extremities of the arc of the meridian to be determined is not actually measured, which would in all cases be laborious, and in most, impossible. A base of four or five miles is first measured with the utmost care, and is connected with the two extremities of the arc of the meridian by a series of triangles, the sides and angles of which are measured or computed; these triangles are in different planes by reason of the inequalities of the surface, and must be reduced by computation to what they would have been, had they been measured on the surface of the sea; they then require a correction to reduce them from plane to spherical triangles. When the length of an arc has thus been found in feet or fathoms, and divided by the degrees and parts of

a degree contained in the difference of latitude of its extreme points, the exact length of one degree is obtained; and from several measurements of this sort it appears that the lengths of the degrees increase from the equator to the pole, and consequently the convexity of the Earth diminishes from the equator to the pole, according to a law which would hold in a spheroid whose equatoreal and polar radii are 3962,8 and 3949,5 miles; their difference, or 13,3 miles, divided by the equatoreal radius, gives $\frac{1}{298{,}33}$ nearly, which fraction is called the *compression* or ellipticity; because according as it is greater or less, the terrestrial spheroid is more or less flattened at the poles.

The above is the simplest and least dependent on theory, of all methods for finding the Earth's ellipticity; all that is required is, the distance of two points nearly on the same meridian and the latitude of each, determined at *two* places on the Earth whose difference of latitude is considerable. There are however two other methods totally independent of one another and of the preceding, viz. the comparison of the length of the seconds pendulum in different latitudes, and the comparison of the lunar inequalities (i. e. the periodic variations in longitude and latitude of the Moon occasioned by the excess of matter at the Earth's equator) as deduced from observation and from theory.

Figure of the Earth from measures of Meridians and Parallels.

131. Having given D and D', the lengths of two degrees AA', BB', of the meridian, (fig. 4.) whose middle points are respectively in latitudes l and l', to find a, c, and $\frac{c}{a}$.

Since an arc of the meridian, the difference of the latitudes of whose extremities is 1^0, may be considered as coincident with the arc of the circle of curvature at its middle point subtending 1^0 at the center of that circle; if m denote the number of degrees in an arc equal to the radius, we have

$$r = mD = a - 2c + 3c \sin^2 l,$$

$$\text{similarly, } mD' = a - 2c + 3c \sin^2 l';$$

$$\therefore\ m\,.\,(D - D') = 3c\,.\,(\sin^2 l - \sin^2 l')\,;$$

$$\therefore\ c = \frac{m\,.\,(D - D')}{3\,(\sin^2 l - \sin^2 l')} = \frac{m\,(D - D')}{3\,.\,\sin(l + l')\,.\,\sin(l - l')}\,;$$

and, substituting this value in the first equation,

$$a = m\,.\,\frac{2\,(D - D') + 3\,(D' \sin^2 l - D \sin^2 l')}{3 \sin(l + l')\,.\,\sin(l - l')}\,;$$

therefore, by division, the compression

$$\frac{c}{a} = \frac{D - D'}{2\,(D - D') + 3\,(D' \sin^2 l - D \sin^2 l')}$$

$$= \frac{D - D'}{3\,(D' \sin^2 l - D \sin^2 l')} \left\{1 + \frac{2\,(D - D')}{3\,(D' \sin^2 l - D \sin^2 l')}\right\}^{-1}$$

$$= \frac{D - D'}{3\,(D' \sin^2 l - D \sin^2 l')}\,,$$

neglecting the square of $D - D'$.

132. If E be the length of a degree at the equator, P at the pole, and M at the mean latitude 45°,

$$mE = a - 2c,$$

$$mP = a + c,$$

$$mM = a - \tfrac{1}{2}c\,;$$

$$\therefore\ M = \tfrac{1}{2}(P + E).$$

The excess of a degree in any latitude above that at the equator,

$$\text{or } D - E \propto \sin^2 l.$$

133. *From any two measured arcs of the meridian, to determine a and c.*

Let Pe (fig. 5.) $= s$, $l =$ circular measure of the latitude, and r as before.

Then (MILLER's *Diff. Cal.* Art. 95.)

$$d_l s = r = a - 2c + 3c\sin^2 l$$

$$= a - \tfrac{1}{2}c - \tfrac{3}{2}c\cos 2l;$$

$$\therefore\ s = al - \tfrac{1}{2}cl - \tfrac{3}{4}.c\sin 2l + C.$$

Let S be the whole arc taken between the values of l' and l;

$$\therefore\ S = a.(l'-l) - \tfrac{1}{2}c.(l'-l) - \tfrac{3}{4}.c(\sin 2l' - \sin 2l)$$

$$= pa + qc,$$

where $p = l' - l$,

and $q = -\tfrac{1}{2}.(l'-l) - \tfrac{3}{4}.(\sin 2l' - \sin 2l)$.

Similarly for another arc S',

$$S' = p'a + q'c;$$

$$\therefore\ a = \frac{q'S - qS'}{pq' - qp'},$$

$$c = \frac{p'S - pS'}{p'q - q'p}.$$

134. From two measured arcs, one of meridian, the other of parallel, to find a and c.

Let Σ be the length of an arc of parallel in latitude λ, and L the circular measure of the difference of longitude of its extremities; then (Art. 126.)

$$\Sigma = L\cos\lambda\,(a + c\sin^2\lambda),$$

and, as before,

$$S = a(l'-l) - \tfrac{1}{2}c(l'-l) - \tfrac{3}{4}c(\sin 2l' - \sin 2l);$$

from which two equations a and c may be determined.

Figure of the Earth from measures of arcs perpendicular to the Meridian.

135. Since a spheroid is a solid of revolution, the normal to the surface at any point will pass through the axis; if, therefore, PR, QR, (fig. 5.) be two consecutive normals

to the very small arc PQ, which is perpendicular to the meridian, they will cut the axis in the same point R; $\therefore$ PR (R) is the radius of curvature of PQ.

$$\text{And } R = PG \cdot \frac{CN}{GN} = n \cdot \frac{a^2}{b^2} = \frac{a}{\sqrt{1 - e^2 \sin^2 l}}$$

$$= a\,(1 + \tfrac{1}{2} e^2 \sin^2 l) = a + c \sin^2 l.$$

Since $R - r = 2c\cos^2 l$, which is always positive, r is less than R, or the curvature of the terrestrial meridian, is greater than that of the perpendicular arc through the same point.

136. Since the parts of the Earth's surface on each side of the meridian are similar, upon the hypothesis of the Earth's being a solid of revolution; therefore of all sections of the surface, made by planes passing through the normal, the meridian must be the section either of greatest or least curvature, and from the preceding Art. is the section of greatest curvature; hence if ρ be the radius of curvature of a section made by a plane passing through the normal inclined at an angle α to the meridian, (Hymers's *Geometry of Three Dimensions*, Art. 222.)

$$\rho = \frac{Rr}{R\cos^2\alpha + r\sin^2\alpha}$$

$$= a + c\sin^2 l - 2c\cos^2\alpha \,.\, \cos^2 l,$$

by substituting the approximate values of R and r, and reducing.

137. Given D, the length of a degree of the meridian at any place, and Δ the length of a degree perpendicular to the meridian, to find a and c.

$$m\Delta = a + c\sin^2 l,$$

$$mD = a - 2c + 3c\sin^2 l;$$

$$\therefore\ m(\Delta - D) = 2c\cos^2 l,$$

$$\therefore\ c = \frac{m\,.\,(\Delta - D)}{2\cos^2 l},$$

$$a = m\,.\,\{\Delta - \tfrac{1}{2}\,.\,(\Delta - D)\,.\,\tan^2 l\};$$

$$\therefore\ \frac{c}{a} = \tfrac{1}{2}\,.\,\frac{\Delta - D}{\Delta\,.\cos^2 l},$$

neglecting the square of $\Delta - D$.

In the same manner, the compression of the terrestrial spheroid may be found, by measuring degrees on arcs inclined at any given angles to the meridian, by means of the formula

$$\rho = a + c\sin^2 l - 2c\cos^2\alpha\,.\cos^2 l.$$

138. *To find the compression by means of a seconds' pendulum, considering the Earth as a spheroid of equilibrium revolving round its minor axis.*

The length of a pendulum oscillating seconds in any latitude varies as the force of gravity at the same place; but considering the Earth heterogeneous, whatever be the law of its density, by Clairaut's Theorem, force of gravity in any latitude $= G(1 + n\sin^2 l)$, (Airy's Tract, Art. 63.), G being the gravity at the equator,

$$\text{and } n = \frac{5m}{2} - \frac{a-b}{b},$$

where m represents the ratio of the centrifugal force at equator to the equatoreal gravity, and has for its numerical value .0034672. Hence, if p, p', be the lengths of two pendulums oscillating seconds in latitudes l, l',

$$\frac{p}{p'} = \frac{1 + n\sin^2 l}{1 + n\sin^2 l'};\ \therefore\ n = \frac{p' - p}{p\sin^2 l' - p'\sin^2 l},$$

$$\text{and then } \frac{a-b}{b} = \frac{5m}{2} - n = .0086679 - n.$$

The value of the compression obtained from observing the intensity of gravity in different latitudes by means of the pendulum, a method of great facility, appears to be about $\frac{1}{290}$, a little greater than that given by measures of arcs.

139. Besides the above modes of determining the compression of the Earth, there is, as stated above, a third method by observations on the Moon. The Moon's motion is subject to several inequalities, which can be determined by observation, or calculation; two of these, which are observed to be $6''$ and $-8''$, are found by calculation to depend on the compression; and the comparison of the calculated and observed inequalities, gives, from the first, a compression $= \frac{1}{305}$, and from the second, $= \frac{1}{304.6}$, which nearly coincide with each other, and with the mean compression deduced from the measurement of degrees. The near agreement of these results is a convincing proof of the truth of the principle of gravitation. If the Earth had been originally a homogeneous fluid mass revolving about an axis in $23^h. 56^m. 4^s$, the compression would have been $\frac{1}{230}$; hence the Earth is not homogeneous; and the Moon not only by her eclipses shews the Earth to be spherical, but by her inequalities determines the deviation from that figure, and discloses a fact relative to the internal structure of the Earth. It must be observed that the difference of the compressions given by the comparison of different arcs is too great to be attributed solely to errors of observation; therefore the Earth is not exactly a solid of revolution.

140. The exact determination of the positions of places on the Earth's surface, is one of the greatest advantages derived from Astronomy; these positions are fixed by latitudes and longitudes as has been explained; the third co-ordinate, viz. the altitude above the level of the sea, is given by the observed height of the barometer. The form of the Earth furnishes a standard of measures for the ordinary purposes of life, as well as for the estimation of the distances and volumes of the heavenly bodies. The length of the pendulum vibrating seconds in the latitude of London in vacuo is 39,13734 inches;

this is the standard of the British measure of extension. The French unit of linear measure is the *metre* which is the ten millionth[*] part of that quadrant of the meridian which passes through Formentara and Greenwich, the middle of which is nearly in the forty-fifth degree of latitude, and is equal to 39,37079 inches. The length of the pendulum can be more easily verified or recovered than that of the metre, and can also be ascertained, it would seem, with superior exactness.

141. In applying the above formulæ to the determination of the compression from the length of two degrees of the meridian in different latitudes, those arcs ought to be chosen, which differ most from each other in situation, as being more likely to cause a compensation of errors, and to give a result less affected by imperfections in the observations. For although the general figure of the meridian, from the pole to the equator may be an ellipse, yet particular parts may differ considerably from it.

The latitudes of the extremities of an arc of the meridian cannot be depended on to more than $1''$, and the deviation of the plumb-line, caused by the attraction of mountains, may sometimes cause much greater errors. From this cause, or from real inequalities, the meridian in some places differs considerably from the meridian determined above.

Geodesic line.

142. The definition of the terrestrial meridian, given in the preceding chapter, is not quite correct, for unless the Earth be a solid of revolution, the meridian will not be a plane curve. The celestial meridian passes through the axis of the Earth and through the zenith of the observer, and all places of the Earth, which have their zeniths in the same

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[*] This was the original definition of the metre; but Bessel (Astron. Nachrichten, No. 333) has shewn that, in that spheroid which most nearly corresponds with the existing measurements of Arcs of the Meridian, the length of the quadrant of the meridian is 10000565·278 metres. He has also, for the same spheroid, obtained the following results expressed in Toises, each of which equals 6·3945925 English feet.

$$\begin{aligned} a &= 3271953\cdot854, \\ b &= 3261072\cdot900, \end{aligned} \quad \frac{a}{a-b} = 300\cdot7047.$$

Degree of meridian, the latitude of the middle point of which is ϕ,

$$= 57011\cdot453 - 284\cdot851 \cos 2\phi + 0\cdot593 \cos 4\phi - 0\cdot001 \cos 6\phi.$$

Degree of parallel $= 57153\cdot885 \cos\phi - 47\cdot576 \cos 3\phi + 0\cdot059 \cos 5\phi.$

celestial meridian, may be considered as having the same terrestrial meridian. Now, considering the immense distance of the stars, the vertical at each of these places may be considered parallel to the plane of the celestial meridian; therefore the *terrestrial meridian* may be defined to be the curve in which the vertical lines at the different points are parallel to the plane of the same celestial meridian, which will be a plane curve only when the Earth is a solid of revolution. The terrestrial meridian, determined by trigonometrical measurement, is not exactly the same as the above, as will appear from the mode of describing it. To measure an arc of the meridian trigonometrically, a horizontal triangle ABC (fig. 6.) is formed, having one angle coinciding with A, the beginning of the arc to be measured $Aabc$, and the other angles with known objects B and C: having determined the length of the first portion Aa, and its direction with respect to the sides of the triangle ABC; to find a second portion ab, a new horizontal triangle BDC is formed, and ab is drawn in the plane BDC, so as to make the same angle with BC that Aa does; in the same manner, the parts ab, bc make equal angles with BD, and so on for the remaining parts of the whole arc to be measured. The triangle ABC is not in the same plane with BDC; therefore, Aa produced must fall above ab, and in the same manner the other small parts of the line Abc, which may be considered tangents to the Earth's surface at their middle points, will, when produced, fall above the adjacent portions. The line thus described is called a Geodesic line, and possesses curious properties; it only coincides exactly with the terrestrial meridian, when the Earth is a solid of revolution; the difference of lengths of these two lines, is, however, so small that it may be safely neglected in any calculations.

CHAPTER III.

TO DETERMINE THE PLANE OF THE MERIDIAN OF ANY PLACE; AND TO COMPUTE THE EFFECT OF THE ERRORS OF INSTRUMENTS FIXED IN THAT PLANE UPON THE TIMES OF TRANSIT OF THE HEAVENLY BODIES.

143. THE principal observations of the heavenly bodies are made as they cross the plane of the meridian, which renders the exact determination of that plane of great importance. The meridian plane may be found, when its intersection with the horizon, or the meridian line is determined, by suspending two plumb-lines over different points of the meridian line; then both of them, being perpendicular to the horizon, are in the plane of the meridian. In the infancy of the science the following method was employed for determining the plane of the meridian.

144. *To draw a meridian line by the shadow of a vertical gnomon on a horizontal plane.*

Let $OHRG$ (fig. 2.) be the plane of the horizon, ZPR that of the meridian, OT a vertical rod, S the Sun on the east side of the meridian in the vertical plane SZO; therefore Ot, determined by joining ST and producing it to meet the plane of the horizon, is the corresponding shadow of the rod, and is equal to $OT \times \tan STZ = OT \times \tan$ Sun's zenith distance, for $\angle STZ$ is the same thing as $\angle SOZ$, the angle which TO subtends at S being quite insensible. With center O and radius Ot describe a circle, and let t' be the point where the extremity of the shadow falls upon the circumference of this circle after the Sun has crossed the meridian; then the length of the shadow being the same, the Sun's zenith distance is the same as before, that is, $ZS = ZS'$; and if his declination be supposed invariable, the co-declinations PS, PS' are equal, and PZ the co-latitude of the place is common to the two triangles, therefore the sides of the spherical triangle SZP

are equal to the sides of $S'ZP$; therefore the Sun's azimuths SZP and $S'ZP$ at the two observations are equal, and their supplements HOR, GOR are equal, or the meridian line OR bisects the angle between the shadows. Hence to draw a meridian line, we have only to bisect the angle between any two equal shadows.

145. The exact vertical position of the gnomon must be ascertained by a plumb-line, and the horizontality of the plane on which the shadow moves, by a level. The objections to this mode of drawing a meridian line are, the difficulty of noting exactly the extremity of the shadow, and the circumstance of the Sun's declination not being invariable, as supposed in the proof. The error arising from the first cause, has been attempted to be removed, by making the extremity of the gnomon globular, and taking the center of its shadow for the extremity of the rod's shadow; or by placing a thin lamina of metal, pierced with a hole at its center, at the end of the rod, and observing where the spot of light falls. All these methods diminish the error arising from the ill-defined shadow of the extremity, and it may be more lessened by taking the mean of several meridian lines determined in this way for the true one.

Determination of the errors of adjustment of a Transit.

146. It was stated in the description of the transit instrument in what manner the three errors of adjustment might be reduced by altering the position of the wire-frame and the supports of the axis; but as it would be attended with great trouble, as well as uncertainty, to perform these adjustments so frequently as to insure the constant accuracy of the instrument, it is preferable to correct the observation for the effects of the errors rather than to attempt their entire removal. It is therefore a principle in observing to regard the instrument as always out of adjustment; and the following propositions have for their object the computation of the effect of the errors of the instrument on the time of transit of the body observed.

147. To determine the effect on the time of a star's transit, of a given small error of level of the axis of a transit instrument.

Let Z (fig. 6. bis) be the zenith, P the pole, $HS'h$ the plane in which the line of collimation of the transit moves in consequence of a given small error of level $(a) = S'HZ$. Let S, S', be the places where a known star crosses the meridian, and the circle $HS'h$; then $S'PS$ is the hour angle corresponding to the difference of the observed and true time of transit.

Let $\delta =$ declination of the star, and $l =$ latitude of the place; then

$$\frac{SPS'}{a} = \frac{\sin SPS'}{\sin a} \text{ nearly}$$

$$= \frac{\sin HS'}{\sin PS'}, = \frac{\sin HS}{\sin PS} \text{ nearly}$$

$$= \frac{\cos (l-\delta)}{\cos \delta} = \cos l + \sin l \tan \delta;$$

∴ error in time of transit in hours

$$= \frac{SPS'}{15} = \frac{a}{15} (\cos l + \sin l \tan \delta),$$

supposing a expressed in degrees.

148. To determine the effect of a given small error of collimation upon the time of a star's transit.

In consequence of this error, the line of collimation traces out a small circle parallel to the meridian. Retaining the same figure and notation as before, and considering $a = SS'$ the error of collimation, we have

$$SPS' = a \sec \delta;$$

∴ error in time of transit $= \dfrac{a}{15} \sec \delta$ hours.

149. To find the azimuthal deviation from the meridian, of a transit instrument, from the observed superior and inferior transits of the same circumpolar star.

Let Z (fig. 11.) be the zenith, P the pole, ZS the vertical plane deviating towards the east in which the line of collimation of the telescope moves; $SS'a'a$ the parallel of a circumpolar star, a being the real and S the observed place where it crosses the meridian at the superior transit, and a', S', those at the inferior transit. Then

$$\frac{\sin S'Pa'}{\sin PZS'} = \frac{\sin S'PZ}{\sin S'ZP} = \frac{\sin S'Z}{\sin PS'}.$$

But since the angles $S'Pa'$, PZS' are very small, their measures may be taken proportional to their sines, and

$$\sin S'Z = \sin Za' \text{ nearly} = \sin(90^\circ - l + 90^\circ - \delta) = \sin(l + \delta);$$

$$\therefore \quad \frac{S'Pa'}{PZS'} = \frac{\sin(l+\delta)}{\cos\delta} = \cos l \tan\delta + \sin l,$$

$$\text{or} \quad S'Pa' = SZP \, . \, (\cos l \tan\delta + \sin l).$$

Similarly, $\dfrac{\sin SPZ}{\sin SZP} = \dfrac{\sin ZS}{\sin PS}$, or, since $ZS = Za$ nearly $= \delta - l$,

$$SPZ = SZP \frac{\sin(\delta - l)}{\cos\delta} = SZP \, . \, (\cos l \tan\delta - \sin l).$$

Let T, T' be the sidereal times of the superior and inferior transits over the meridian, t and t' the times of observation at S and S' (the former of which precedes the transit and the latter follows it) all expressed in degrees;

$$\therefore \quad T' = t' - SZP \, . \, (\cos l \, . \tan\delta + \sin l);$$

$$T = t + SZP \, . \, (\cos l \tan\delta - \sin l);$$

$$\therefore \quad T - T' = t - t' + 2\, SZP \, . \cos l \tan\delta.$$

Now $T - T' = 12^h$ or 180°;

$$\therefore \quad SZP = \tfrac{1}{2} \, . \, \frac{180^\circ - (t - t')}{\cos l \, . \tan\delta},$$

which is the deviation towards the east, of the plane in which the line of collimation moves. This result depends only on the latitude of the place, the observed interval of the superior and inferior transits, and the star's declination, and does not require the difference of right ascension; but as the transits take place after an interval of twelve hours, one of them for the greater part of the year must be observed by day-light; and the stars near the equator, the time of whose transits can be observed most accurately from the rapidity of their motions, cannot be used. Hence for portable instruments the following method of determining the meridian error is sometimes employed.

150. *When the vertical plane in which a transit instrument moves, nearly coincides with the meridian, to find the deviation, from the observed times of transit of two known stars.*

Let Sa (fig. 11.) be the parallel of a known star, then using the same construction and process as in the former case, we have

$$\therefore\ SPZ = SZP \,.\, (\cos l \,.\, \tan\delta - \sin l).$$

Let T be the sidereal time of the star's transit over the meridian, t the time of the observation at S, both expressed in degrees at the rate of 15^0 to an hour.

$$\text{Then}\quad T = t + SZP \,.\, (\cos l \,.\, \tan\delta - \sin l).$$

Similarly, if T', t', δ' are values of T, t, δ for another star,

$$T' = t' + SZP \,.\, (\cos l \,.\, \tan\delta' - \sin l);$$

$$\therefore\ T - T' = t - t' + SZP \,.\, \cos l \,.\, (\tan\delta - \tan\delta')$$

$$= t - t' + SZP \,.\, \cos l \,.\, \frac{\sin(\delta - \delta')}{\cos\delta \,.\, \cos\delta'};$$

$$\therefore\ SZP = \{T - T' - (t - t')\} \,.\, \frac{\cos\delta \,.\, \cos\delta'}{\cos l \,.\, \sin(\delta - \delta')}.$$

In this equation $T - T'$ the difference of the right ascensions of the two stars, is known from the catalogues in which all stars are registered with their right ascensions and declinations, and $t - t'$ may be observed; therefore PZS, the deviation of the transit instrument from the meridian, may be found and corrected.

In order that the observed error $T - T' - (t - t')$, caused by the deviation of the transit instrument, may be the greatest possible, those stars ought to be chosen, in which $\delta - \delta' = 90^0$, nearly; so that if the one star cross the meridian near the zenith, the other ought to cross it near the horizon, or if the declination of the one star is small, that of the other ought to be nearly 90^0.

151. In dialling, where the plane of the meridian is required, it is usual to find it by fixing a plate of metal at a convenient distance from a plumb-line, and in such a position that two stars whose right ascensions are nearly equal or differ by 180^0 when viewed through a small hole in the plate are bisected by the plumb-line; then the plane passing through the hole and the plumb-line nearly coincides with the meridian; and true noon happens nearly at the instant when the Sun's rays, passing through the hole, fall upon the plumb-line.

The pole star, and ϵ in the tail of the great bear, whose difference of right ascension $= 177^0.43'.11''$; or α Ophiuchi, and β Draconis, where the difference of right ascension is $7''$, or the pole star and γ Cassiopeiæ, are adapted for finding the meridian by this method.

152. To find the effect on the time of transit of an observed body, considering the errors of adjustment of a Transit Instrument coexistent.

Let P (fig. 7. bis) be the pole, Z the zenith of the place of observation, E the East point of the horizon, L the East end of the diameter of the sphere parallel to the axis of the transit; S a star on the meridian, T the same star on the mean wire of the transit, h its hour angle $= TPS = TPL - ZPL$, then $\frac{h}{15}$ will be the time when the star is on the middle wire $-$ time of star's culmination. Join LT, LZ, LP, by arcs of great circles, which, from the errors of adjustment being small, are nearly quadrants; join SP, then observing that when an angle (a) is nearly a right angle, $\sin a = 1$, $\cos a = \frac{\pi}{2} - a$, and that $SP = TP$ nearly, we get

$$\cos TL = \cos PT \cos PL + \sin PT \sin PL \cos TPL,$$

$$\frac{\pi}{2} - TL = \cos SP\left(\frac{\pi}{2} - PL\right) + \sin SP\left(\frac{\pi}{2} - TPL\right),$$

$$\text{but } h = TPL - ZPL = \frac{\pi}{2} - ZPL - \left(\frac{\pi}{2} - TPL\right);$$

$$\therefore h = \frac{\pi}{2} - ZPL + \operatorname{cosec} SP . \left(TL - \frac{\pi}{2}\right) + \cot SP\left(\frac{\pi}{2} - LP\right),$$

which is the error in hour angle expressed by the deviations in hour angle and declination of the axis, the error of collimation, and the declination of the star.

Again, $\cos LZ = \cos ZP \cos PL + \sin ZP \sin LP \cos LPZ$;

$$\therefore \frac{\pi}{2} - LZ = \cos ZP\left(\frac{\pi}{2} - PL\right) + \sin ZP\left(\frac{\pi}{2} - LPZ\right);$$

substitute this value of $\frac{\pi}{2} - LPZ$ in the above;

$$\therefore h = \left(\frac{\pi}{2} - LZ\right)\operatorname{cosec} ZP + \left(\frac{\pi}{2} - LP\right)(\cot SP - \cot ZP)$$

$$+ \left(TL - \frac{\pi}{2}\right)\operatorname{cosec} SP,$$

$$\text{or } \sin PZ \sin SP . h = \left(\frac{\pi}{2} - LZ\right)\sin SP + \left(LP - \frac{\pi}{2}\right)\sin ZS$$

$$+ \left(TL - \frac{\pi}{2}\right)\sin ZP,$$

where the error in hour angle is expressed by the deviations in zenith distance and declination of the axis.

Again, $\cos SL = \cos SZ \cos LZ + \sin SZ \sin LZ \cos SZL$,

$$\text{or, } \frac{\pi}{2} - SL = \cos SZ\left(\frac{\pi}{2} - LZ\right) + \sin SZ\left(LZP - \frac{\pi}{2}\right),$$

$$\text{and} \quad \sin SP \,.\, h = TS = TL - \frac{\pi}{2} + \frac{\pi}{2} - SL\,;$$

$$\therefore \quad \sin SP \,.\, h = TL - \frac{\pi}{2} + \cos SZ \left(\frac{\pi}{2} - LZ\right) + \sin SZ \left(LZP - \frac{\pi}{2}\right),$$

where the error in hour angle is expressed in terms of the errors in level and azimuth of the axis.

CHAPTER IV.

TO FIND THE LATITUDE OF A PLACE ON THE EARTH'S SURFACE.

153. The latitude of a place on the Earth's surface may be found in various ways, and the mode to be adopted in any particular case must be determined by the observations which it is possible to make, from the nature of the instruments and the opportunities of using them.

To find the latitude by means of a meridian circle.

1. If the Sun's declination is known, observe the zenith distance of the Sun's center as he crosses the meridian; let this be ZC (fig. 1.) and let QE be the equator; then $ZQ = QC + ZC$, or the latitude = the declination + the Sun's meridian zenith distance.

2. Again, if a known star be observed which crosses the meridian at C or L,

$$ZQ = QC + CZ, \text{ or } = QL - LZ,$$

that is, the latitude = the declination ± the observed meridian zenith distance of a known star; + if the star crosses the meridian between the zenith and equator, and − if the star crosses the meridian between the zenith and pole.

3. If two altitudes A and a, of the same circumpolar star, whose parallel is Ll, be observed at the times of its superior and inferior transit over the meridian,

$$\text{the latitude} = \text{the altitude of the pole} = LH - LP,$$

$$= A - \text{the star's north polar distance.}$$

Similarly,

$$\text{the latitude} = lH + lP = a + \text{the star's north polar distance};$$

$$\therefore \text{ the latitude} = \tfrac{1}{2}.(A + a), \text{ by addition.}$$

These methods may be used with any instrument adapted to observing angles in the plane of the meridian. Or, if the exact time of the Sun or star's being on the meridian is known, they may be used with an instrument for taking altitudes, not fixed in the plane of the meridian, in which latter case they are applicable to finding the latitude at sea.

154. *To find the latitude from two equal altitudes of the Sun, observed before and after noon.*

Let Z (fig. 9.) be the zenith, P the pole, S and S' the two positions of the Sun at the equal zenith distances ZS, ZS'; then, if the Sun's declination, which changes very slowly, (about 23° in three months) be considered the same at the two observations, $PS = PS'$, and

$$\angle SPZ = S'PZ = \tfrac{1}{2}.(SPS').$$

Now if t be the sidereal hours by the clock between the observations, and T the sidereal hours between the preceding and succeeding noon, considering the Sun's motion in right ascension uniform, and consequently his retarded (i. e. with respect to the fixed stars) diurnal motion uniform, during that interval,

$$SPS'\ (2\alpha) = 360^0\frac{t}{T},$$

which determines SPZ (α); and from this angle, the Sun's declination $(90^0 - PS)$, and the observed altitude $(90^0 - ZS)$, PZ the co-latitude may be found.

$$\text{Let } PZ = 90^0 - l, \quad PS = 90^0 - \delta, \quad ZS = 90^0 - a,$$

$$\text{then } \sin a = \cos l . \cos \delta . \cos \alpha + \sin l . \sin \delta.$$

Let ϕ be a subsidiary angle, determined by the equation

$$\tan \phi = \cos \alpha . \cot \delta;$$

$$\therefore \sin a = \cos l . \sin \delta . \tan \phi + \sin l . \sin \delta$$

$$= \sin \delta . \frac{\cos l . \sin \phi + \sin l . \cos \phi}{\cos \phi}$$

$$= \frac{\sin \delta . \sin (l + \phi)}{\cos \phi};$$

$$\therefore \sin(l + \phi) = \frac{\sin a \cdot \cos \phi}{\sin \delta},$$

which determines $l + \phi$, and, therefore, the latitude l.

155. In the same manner, the latitude may be found by two equal altitudes of the same star.

A mode of correcting the error in the determination of the angle ZPS, introduced by supposing the Sun's declination invariable, will be given in the Chapter on Time.

The only use of the *two* altitudes is to determine the angle SPZ; but if the time (t) between the first observation and the instant of the Sun's passage over the meridian be known, that angle is found at once by the equation

$$SPZ = 360^0 \frac{t}{T},$$

and one altitude will then suffice for determining the latitude.

As the Sun may be obscured by clouds at the instant when the second observation ought to be made, the following method of determining the latitude by *any* two observed altitudes is frequently useful.

156. *To find the latitude and hour angle, from two altitudes of the Sun, and the time between.*

Let Z (fig. 12.) be the zenith, P the pole, S and S' the two positions of the Sun's center; then, from the time between the observations, SPS' (a) may be found as in the preceding Article.

Let $ZS = z$, $ZS' = z'$, which are the complements of the observed altitudes,

$PS = \Delta$, $PS' = \Delta'$, the co-declinations of S and S',

$SS' = \theta$, and $l = 90^0 - PZ$, the latitude of Z.

Then, in the triangle PSS',

$$\tan \tfrac{1}{2}(PSS' + PS'S) = \frac{\cos \frac{1}{2}(\Delta' - \Delta)}{\cos \frac{1}{2}(\Delta' + \Delta)} \cdot \cot \frac{a}{2},$$

$$\tan \tfrac{1}{2}(PSS' - PS'S) = \frac{\sin \frac{1}{2}(\Delta' - \Delta)}{\sin \frac{1}{2}(\Delta' + \Delta)} \cdot \cot \frac{a}{2};$$

$\therefore \angle\, PSS' = \frac{1}{2}.(PSS' + PS'S) + \frac{1}{2}.(PSS' - PS'S)$, is known,

and consequently $\angle\, \theta$ from the equation $\sin\theta = \dfrac{\sin\Delta'.\sin\alpha}{\sin PSS'}$.

Again in the triangle ZSS',

$$\sin\tfrac{1}{2}(ZSS') = \sqrt{\frac{\sin\frac{1}{2}(z'+z-\theta).\sin\frac{1}{2}(z'+\theta-z)}{\sin\theta.\sin z}},$$

and $\angle\, PSZ = PSS' - ZSS'$, are known.

Lastly, in the triangle ZPS,

$$\sin l = \cos z.\cos\Delta + \sin z.\sin\Delta.\cos PSZ,$$

and $\sin ZPS = \sin ZSP.\dfrac{\sin z}{\cos l}$, which give the latitude and hour angle.

The last equation but one may be replaced by the formulæ

$$\tan\phi = \cos PSZ.\tan\Delta, \text{ which determines } \phi,$$

and $\sin l = \cos\Delta(\cos z + \sin z\tan\phi) = \dfrac{\cos(z-\phi).\cos\Delta}{\cos\phi}$, which gives l, both of which are adapted to logarithmic calculation.

Thus these equations determine successively PSS', θ, ZSS', ϕ, and, lastly, l, and ZPS, which are the quantities required.

The figure represents the two positions of the Sun on the same side of the meridian, but the proof is the same when they are on different sides.

157. The Sun's declination, which is required in the above method, is subject to daily alteration, and is registered in tables which contain the declinations corresponding to the noon of every day under a given meridian. From these registered declinations, the declination for any intermediate time under the same meridian is found, by taking a part of the whole increment of declination proportional to the time from the preceding noon; thus if δ and δ' be two declinations from the tables for the times of the preceding and succeeding noon under the given meridian, and T^h the time between the two noons, then $\delta' - \delta$ is the increment of declination in T^h, therefore, in t^h from

the time when the declination was δ, the increment of declination is $\frac{t}{T}\cdot(\delta'-\delta)$, and the whole declination $=\delta+\frac{t}{T}\cdot(\delta'-\delta)$. If, therefore, at the place where the observations are made, we have a chronometer which indicates the time at the place for which the tables are calculated, the declinations, and, therefore, the co-declinations Δ and Δ', may be found. This remark will be better understood after the perusal of the Chapter on Time.

158. In the same manner, the latitude may be found from the observed altitudes of two known stars, and the time between. If the observations are contemporary, $a =$ the difference of the stars' right ascensions (d); if they are separated by an interval of t sidereal hours, supposing the second observed star S' to have been at S_1 at the time of the 1st observation, (fig. 12),

$$a = SPS_1 + S_1PS' = d + 15^0.t;$$

if the second observed star is east of the other, $a = d \sim 15^0.t$.

159. If the Sun's declination be considered invariable between the observations, or, if the latitude be deduced from two altitudes of the same star, the four first equations of Art. 156, reduce themselves to

$$\tan PSS' = \frac{\cot\frac{1}{2}a}{\cos\Delta},$$

$$\sin\tfrac{1}{2}\theta = \sin\tfrac{1}{2}a.\sin\Delta,$$

which are obtained at once from Napier's rules, by dividing the isosceles triangle PSS' into two equal right-angled triangles by means of a perpendicular PO.

The remainder of the solution is the same as before.

In this manner the general problem is often solved; the north polar distance is supposed to remain equal to $\frac{1}{2}(\Delta+\Delta')$ between the observations, and the error of from $1'$ to $5'$, thus introduced, is got rid of by a subsequent correction.

160. When the latitude is nearly known, a more correct value may often be obtained from two altitudes of the Sun, and the time between, by the following method, which is called, from its inventor, *Douwe's Method*, and is much used at sea.

With the same assumptions as before, let l' be the approximate latitude, h and h' the hour angles, and Δ the Sun's north polar distance, which is here taken equal to half the sum of the polar distances at the two observations, and is supposed to remain constant.

$$\text{Then } \cos z = \cos l \sin \Delta \cos h + \sin l \cos \Delta \ldots (1)$$

$$= \cos l' \sin \Delta \cos h + \sin l' \cos \Delta \text{ nearly,}$$

$$\text{and } \cos z' = \cos l' \sin \Delta \cos h' + \sin l' \cos \Delta ;$$

$$\therefore \cos z - \cos z' = \cos l' \sin \Delta . (\cos h - \cos h') ;$$

$$\therefore \sin \tfrac{1}{2} (z' - z) . \sin \tfrac{1}{2} (z' + z)$$

$$= \cos l' . \sin \Delta . \sin \tfrac{1}{2} (h' - h) . \sin \tfrac{1}{2} (h' + h).$$

In this equation, $h' - h$ is known from the observed interval, when the observations are made on the same side of the meridian, and $h' + h$, when they are made on different sides; and either of these being known, the other may be deduced from it by the equation, and thus h may be found.

But from equation (1),

$$\cos z = \sin (l + \Delta) - 2 \cos l \sin \Delta \sin^2 \tfrac{1}{2} h ;$$

$$\therefore \sin (l + \Delta) = \cos z + 2 \cos l' \sin \Delta \sin^2 \tfrac{1}{2} h \text{ nearly,}$$

using the approximate value in the small term; from which $l + \Delta$, and, therefore, l may be found.

If the subsidiary angle ϕ be determined from the equation

$$\cot \phi = \frac{2 \cos l' \sin \Delta}{\sin z} . \sin^2 \tfrac{1}{2} h,$$

$$\text{then } \sin (l + \Delta) = \frac{\sin (z + \phi)}{\sin \phi},$$

which is adapted to logarithmic computation.

If the value of l thus deduced differs much from the approximate value l', the operation must be repeated with l instead of l', in which case the method becomes longer than the direct method, without being so certain. See Delambre, Chap. xxxvi.

161. When the latitude is found *at sea*, from two altitudes of the Sun and the time between, a correction must be applied for the ship's change of place between the observations.

Let P (fig. 13.) be the pole, Z the zenith and S the Sun at the first observation, Z' and S' at the second. Let $SZZ' = m$, and angle subtended by $ZZ' = n$; then m is known from the direction of the ship's motion, and if s be the actual space moved over on the Earth's spherical surface, which is known from the ship's rate and the time, $n = \frac{s}{r}$, r being the Earth's radius. Let z, z', be the observed zenith distances of S and S', and $z_1 = Z'S$ the zenith distance which would have been observed, if the zenith at the first observation had been Z'. Let $z_1 = z + y$, then, from the triangle $Z'ZS$,

$$\cos(z + y) = \sin n \sin z \cos m + \cos z \cos n,$$

or, substituting for $\cos y$ and $\cos n$,

$$\cos z\left(1 - 2\sin^2\frac{y}{2}\right) - \sin z \sin y$$

$$= \sin n \sin z \cos m + \cos z\left(1 - 2\sin^2\frac{n}{2}\right),$$

$$\text{or } \sin y\left(1 + \cot z \tan\frac{y}{2}\right) = \sin n\left(- \cos m + \cot z \tan\frac{n}{2}\right).$$

But since y and n are very small, substituting the circular measures of these angles instead of their sines and tangents,

$$y\,(1 + \tfrac{1}{2} y \cot z) = n\,(- \cos m + \tfrac{1}{2} n \cot z);$$

therefore, for a first approximation, $y = - n \cos m$;

$$\therefore\ y = n\,(- \cos m + \tfrac{1}{2} n \cot z)\,(1 - \tfrac{1}{2} y \cot z)$$

$$= n\,(- \cos m + \tfrac{1}{2} n \cot z)\,(1 + \tfrac{1}{2} n \cot z \cos m)$$

$$= n\left\{- \cos m + \tfrac{1}{2} n \cot z - \tfrac{1}{2} n \cot z\,(\cos m)^2\right\}$$

$$= - n \cos m + \frac{n^2}{2} \cot z\,(\sin m)^2,$$

$$\text{but } z_1 = z + y \text{ and } n = \frac{s}{r},$$

therefore, expressing the formula in seconds*,

$$z_1 = z - \cos m \frac{s}{r \sin 1''} + \tfrac{1}{2} \cot z \, (\sin m)^2 \frac{s^2}{r^2 \sin 1''}.$$

Having thus found z_1, we may proceed, as before, to find the latitude of the ship, at the second observation, from the zenith distances z_1, z', and the time between.

162. *To determine the latitude from zenith distances of a heavenly body, observed very near the meridian.*

Let $z = ZS$ (fig. 10.) be the zenith distance of the Sun or of a known star S, observed very near the meridian, $l = 90^0 - PZ$ the latitude required, $\delta = 90^0 - PS$ the star's known declination, and $h = ZPS$ the hour angle.

Then $\cos z = \cos l \, . \cos \delta \, . \cos h + \sin l \sin \delta$.

But if $m = l - \delta$ be the star's meridian zenith distance,

then $z = m + y$ where y is very small;

$$\therefore \cos (m + y) = \cos l \cos \delta \cos h + \sin l \sin \delta;$$

therefore, expanding and putting for $\cos y$, $\cos h$, their values in terms of $\sin \frac{y}{2}$, $\sin \frac{h}{2}$,

$$\cos m \left(1 - 2 \sin^2 \frac{y}{2}\right) - \sin m \sin y$$

$$= \cos l \cos \delta \left(1 - 2 \sin^2 \frac{h}{2}\right) + \sin l \sin \delta,$$

$$= \cos m - 2 \cos l \cos \delta \sin^2 \frac{h}{2},$$

$$\text{or } \sin y \left(1 + \cot m \tan \frac{y}{2}\right) = \frac{2 \cos l \cos \delta}{\sin m} \, . \sin^2 \frac{h}{2}$$

$$= x \text{ suppose};$$

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* If a'' be the number of seconds in an angle whose circular measure is a; then since $\sin 1''$ is very nearly the circular measure of the angle $1''$, we have

$$a'' = \frac{a}{\sin 1''}, \quad a = a'' \sin 1''.$$

therefore, since y is very small, putting y for $\sin y$, and $\frac{y}{2}$ for $\tan\frac{y}{2}$, we have

$y = x\left(1 - \frac{y}{2}\cot m\right)$; $\therefore$ for a first approximation $y = x$,

and $\therefore$ more nearly, $y = x\left(1 - \frac{x}{2}\cot m\right) = x - \frac{x^2}{2}\cot m$;

or, replacing x and m by their values, and expressing the formula in seconds,

$$y = \frac{2\cos l . \cos\delta . \sin^2\frac{1}{2}h}{\sin 1'' . \sin(l-\delta)}$$

$$-\tfrac{1}{2} . \cot(l-\delta) . \sin 1'' . \left\{\frac{2\cos l . \cos\delta . \sin^2\frac{1}{2}h}{\sin 1'' . \sin(l-\delta)}\right\}^2 + \&c.$$

In calculating the coefficient of $\sin^2\frac{1}{2}h$, l may be taken equal to the sum of the least observed zenith distance and δ. To find h itself, if a transit instrument be used, the clock will shew the interval between the time of observation and the transit, from whence h may be found, as in Art. 155. If there be no transit instrument, an equal zenith distance must be observed on the other side of the meridian, from which h may be calculated, as in Art. 154.

Having by these means got the correction y, $z - y$, the meridian zenith distance is found. If, therefore, several meridian zenith distances of the same star be taken near the meridian, and we call Z the mean of the values of z, and Y of y,

the true meridian zenith distance $= Z - Y$,

and the true latitude $= \delta \pm (Z - Y)$.

163. In general the first term of the value of y is sufficient, and the second, which is easily deduced from it, need not be resorted to, unless the star be at a considerable distance from the meridian. To facilitate the computation, tables are formed of the values of $\frac{2\sin^2\frac{1}{2}h}{\sin 1''}$, and $\left(\frac{2\sin^2\frac{1}{2}h}{\sin 1''}\right)^2$, for all values of h in seconds up to half a degree.

The pole star, which is a star of the first magnitude very near the north pole, on account of its slow motion, may be employed with greater advantage than any other in determining the latitude by this method; with this star, y is very small for more than a quarter of an hour before and after the star's transit.

This method, by which the latitude is found with very great accuracy, may be used by land or sea; in the latter case, h must be determined as in Art. 154.

Taking only the first term of the value of y,

$$y \text{ or } z - m = \frac{2 \cos l \cos \delta}{\sin m} \sin^2 \frac{h}{2};$$

therefore the change of altitude, or zenith distance, near the meridian, varies as $\sin^2 \frac{1}{2} h$, and therefore as h^2 nearly, or as the square of the interval between the time of observation, and the time of transit.

164. To find the latitude of a place by observations of the pole star out of the meridian.

The pole star, on account of the smallness of its north polar distance ($1^\circ . 38' . 47''$), may be advantageously used for finding the latitude, in any point of its circular course.

Let Δ be the apparent polar distance of the star, and let z, a zenith distance corresponding to the hour angle h, $= c + y$, where c is the co-latitude, and y is always very small, and in no case greater than $\pm 1^\circ . 38' . 47''$ (Δ); then

$$\cos (c + y) = \sin \Delta \sin c \cos h + \cos \Delta \cos c;$$

therefore, as in Art. 161,

$$y = - \Delta \cos h + \tfrac{1}{2} \Delta^2 \cot c (\sin h)^2 \text{ nearly};$$

$$\therefore\ c \text{ the co-latitude} = z - y = z + \Delta \cos h - \tfrac{1}{2} \Delta^2 \cot c (\sin h)^2.$$

For a first value, c may be taken $= z + \cos h\, \Delta$, from which the coefficient of Δ^2 may be calculated; therefore, determining n co-latitudes c, c', &c., in this manner, the true co-latitude

$$= \frac{1}{n} . (c + c' + \&\text{c.})$$

Hence if l be the latitude, a the true altitude, s the sidereal time of observation, α the right ascension of the star, so that $h = s - \alpha$, and p the number of seconds in Δ, we have

$$l = a - p \cos h + \tfrac{1}{2} \sin 1'' \, (p \sin h)^2 \tan l.$$

The value of the second term is given in the Nautical Almanac for every 10th minute of sidereal time (assuming as mean values $p = 93'.\ 6''$, $\alpha = 15^0.\ 27'$) and being added to or subtracted from a, as the case may be, gives an approximate latitude; and the value of the third term is given for every half hour of sidereal time and for every 5th degree of latitude, and must be taken out with the approximate latitude. There is, besides, a third correction given in the Nautical Almanac depending upon the difference between the true and assumed values of p and α.

The observations required in Articles 162 and 164 may be made with any instrument for taking altitudes or zenith distances.

CHAPTER V.

ON THE SUN'S MOTION IN THE ECLIPTIC.

165. *To find, by observation, the Sun's motion in right ascension in any time.*

Let t and t', the respective times of transit of the Sun and a fixed star, be observed by the clock; then, if the clock gain or lose uniformly, and $24 + x$ be the hours between two successive transits of the same fixed star, the difference of right ascension of the Sun and star, at the instant the Sun was on the meridian $= 360^0 . \dfrac{t' - t}{24 + x}$.

If A and a be the differences of right ascension of the Sun and the same fixed star, thus found at different times of the year, $a - A$ is the Sun's motion in right ascension in the interval, supposing the star's right ascension to be invariable.

In thus finding the Sun's motion in right ascension, it is not necessary that the hand of the clock should go round exactly in 24 sidereal hours, but merely that the clock should gain or lose time uniformly during the same day, and that the whole gain or loss be known.

166. The stars, as was noticed in the Introduction, and as will be further shewn in the chapters on Aberration and Precession, do not always appear in exactly the same place, but have small apparent changes of position, which can be ascertained; if, therefore, S be the right ascension of the star at the first observation, and $S + e$ at the second,

$A =$ the Sun's right ascension at the first observation $- S$,

$a =$ . . . . second . . . . $- (S + e)$;

$\therefore\ a - A =$ the Sun's motion in right ascension $- e$;

$\therefore$ the Sun's motion in right ascension $= a - A + e$.

167. *To find the Sun's declination, by observation.*

Let the zenith distance of the Sun, when on the meridian, be observed, then, in north latitudes, the observed zenith distance = the latitude of the place ∼ the Sun's declination; therefore the Sun's declination = the latitude of the place ± the observed zenith distance. If the declination, thus found, is negative, the Sun is on the south side of the equator.

In observing the altitude of the Sun, or of any other heavenly body which has apparent magnitude, the altitudes of the upper and lower limb are observed, and the semi-sum of these is taken for the altitude of the center; in the same way to find the time of transit, the times of transit of the western and eastern limb are observed, and their semi-sum taken for the true time of the transit of the center. One observation of the limb will suffice if the Sun's semidiameter be known.

168. If the Sun's declination be thus found on every day throughout the year, it is found to be 0 at one time, and afterwards to increase up to $23^\circ.28'$, at the end of three months; from this time it decreases, and is again 0, at the end of half a year, when the declination becomes south, and the same changes of south declination recur for the next half year. The cause of this apparent change in the Sun's declination we have seen (Art. 46.) to exist, in the Earth's motion in the ecliptic, and the parallelism of the inclined axis of rotation. If the Sun's motion in right ascension (A) in the interval between his declination being 0 and D, be observed, it is found that $\sin A \propto \tan D$, which proves that the Sun's motion is in a *great circle* cutting the equator in the points where his declination is 0. For let ♈ be the point in the equator where his declination is 0 (fig. 14.) and ♈D, DS, ♈d, dS' corresponding right ascensions, and declinations; join ♈S by the arc of a great circle, meeting dS' in s; ∴ from the right-angled triangles ♈DS, ♈ds,

$$\cot S\,\text{♈}\,D = \frac{\sin \text{♈}\,D}{\tan DS} = \frac{\sin \text{♈}\,d}{\tan ds}; \text{ but } \frac{\sin \text{♈}\,D}{\tan DS} = \frac{\sin \text{♈}\,d}{\tan dS'};$$

∴ $\tan dS' = \tan ds$, or $dS' = ds$, each being less than 90°; that is, the Sun is always found in the great circle ♈S.

169. *To find the right ascension of the Sun, and the obliquity of the ecliptic.*

Let ♈ D, (fig. 14.) be the equator, ♈ S the ecliptic, or great circle in which the Sun moves, S and s two observed positions of the Sun, when on the meridian of the observer, $SD(\delta)$, and $sd\,(\delta')$ the corresponding declinations. Let $Dd\,(2a)$ the difference of right ascension of S and s be observed; then if ♈ $D = x$, and S ♈ D the angle between the ecliptic and equator, which is called the obliquity, $= \omega$,

$$\sin x = \tan\delta \,.\, \cot\omega,$$

$$\sin(x + 2a) = \tan\delta' \cot\omega;$$

$$\therefore \frac{\sin(x+2a) + \sin x}{\sin(x+2a) - \sin x} = \frac{\tan\delta' + \tan\delta}{\tan\delta' - \tan\delta}.$$

$$\text{Or } \frac{\tan(x+a)}{\tan a} = \frac{\sin(\delta' + \delta)}{\sin(\delta' - \delta)};$$

$$\therefore \tan(x+a) = \frac{\sin(\delta' + \delta)}{\sin(\delta' - \delta)} \,.\, \tan a,$$

$$\text{and } \cot\omega = \sin x \,.\, \cot\delta,$$

which equations determine x and ω. One of the observations must be made near an equinox, and the other near a solstice.

The value of the obliquity thus found $= 23^{0}.27'.36''$.

170. Having found the right ascension of the Sun, that of *any other heavenly body* may be obtained by adding to it the difference of right ascension of the body and the Sun, determined as in Art. 165.

The right ascension of the Sun, when on the meridian, converted into time at the rate of 15^{0} to an hour, will give the sidereal time from the first point of Aries being on the meridian, that is, the time shewn by the sidereal clock.

171. *Flamstead's method of finding the Sun's right ascension.*

Let ♈ DL (fig. 14.) be the equator, ♈ SL the ecliptic, S the Sun *near* the equinox ♈, when on the observer's meridian,

D his place referred to the equator by the declination circle SD. Let m and n be places of D similarly determined near the equinox L, where the Sun's declinations δ' and δ'' at succeeding noons, are one greater, and the other less than SD (δ), and let D' be the intermediate place where the declination $= \delta$. Then Dm (2α), the difference of the Sun's right ascensions at D and m, may be found (Art. 165), and mn (2β) the difference at m and n.

Now the changes of right ascension and declination may be considered uniform during the same day;

$$\therefore \frac{mD'}{mn} = \frac{\text{time from } m \text{ to } D'}{\text{time between succeeding noons}} = \frac{\delta' - \delta}{\delta' - \delta''};$$

$$\therefore mD' = 2\beta \cdot \frac{\delta' - \delta}{\delta' - \delta''}.$$

Let A be the Sun's right ascension at the first observation;

$$\therefore A + 2\alpha \text{ is the right ascension of } m,$$

$$\text{and } A + 2\alpha + 2\beta \cdot \frac{\delta' - \delta}{\delta' - \delta''} \text{ of } D', \text{ which equals } 180^0 - A;$$

$$\therefore A = 90^0 - \alpha - \beta \cdot \frac{\delta' - \delta}{\delta' - \delta''}.$$

The differences $\delta' - \delta$, $\delta' - \delta''$ are equal to the differences of the observed meridian zenith distances, which renders the knowledge of the latitude not requisite in Flamstead's method, and causes all errors which equally affect the observed zenith distances to vanish. These advantages render the approximate method in practice more accurate than the preceding, which is theoretically exact. The observations necessary for finding the Sun's right ascension by this method are separated by an interval of nearly half a year; they are made near the equinoxes because then the changes in declination are most rapid.

172. When the Sun is in one of the solstices, the right ascension $= 90^0$, and the declination is equal to the obliquity (ω). If, therefore, we suppose the Sun (☉) to be on the meridian of any place whose latitude is (l), at the time when he is in the

solstices, and that the altitudes A and A' are observed corresponding to the summer and winter solstice,

$$A = 90^0 - \odot\text{'s zenith dist.} = 90^0 - (l - \odot\text{'s dec.}) = 90^0 - l + \omega.$$

Similarly, $A' = 90^0 - \odot\text{'s zenith dist.} = 90^0 - (l + \odot\text{'s dec.}) = 90^0 - l - \omega$;

$$\therefore\ \omega = \tfrac{1}{2}.(A - A'),$$

$$l = 90^0 - \tfrac{1}{2}.(A + A').$$

173. *To find the relation between the Sun's right ascension, declination, and longitude.*

Let ♈S (fig. 14.) the Sun's longitude $= l$, ♈D the right ascension $= a$, SD the declination $= \delta$, then applying Napier's rules to the right-angled triangle ♈SD,

$$\tan a = \cos\omega \,.\, \tan l \ \ldots\ldots\ldots\ (1),$$

$$\tan\delta = \sin a \,.\, \tan\omega \ \ldots\ldots\ldots\ (2),$$

$$\sin\delta = \sin\omega \,.\, \sin l \ \ldots\ldots\ldots\ (3),$$

$$\cos l = \cos a \,.\, \cos\delta \ \ldots\ldots\ldots\ (4);$$

which equations, when two of the quantities a, ω, δ, l are given, determine the others.

174. The equation

$$\tan a = \cos\omega \tan l, \quad \text{or } \tan l = \frac{1}{\cos\omega}\tan a,$$

admits of useful transformations; substituting in it the exponential values of tan a, tan l, and proceeding as in Art. 127, we obtain,

$$a = l - \tan^2\tfrac{1}{2}\omega \,.\, \sin 2l + \tfrac{1}{2} \,.\, \tan^4\tfrac{1}{2}\omega \,.\, \sin 4l - \&\text{c.}$$

$$l = a + \tan^2\tfrac{1}{2}\omega \,.\, \sin 2a + \tfrac{1}{2} \,.\, \tan^4\tfrac{1}{2}\omega \,.\, \sin 4a + \&\text{c.}$$

$$\therefore\ l - a = \tan^2\tfrac{1}{2}\omega \,.\, \sin 2l - \tfrac{1}{2} \,.\, \tan^4\tfrac{1}{2}\omega \,.\, \sin 4l + \&\text{c.}\ldots(\alpha),$$

and $l - a = \tan^2\tfrac{1}{2}\omega \,.\, \sin 2a + \tfrac{1}{2} \,.\, \tan^4\tfrac{1}{2}\omega \,.\, \sin 4a + \&\text{c.}\ldots(\beta)$.

These series are called the *reductions of the ecliptic to the equator*, and are useful in calculating tables.

175. Proceeding in the same manner with the equation

$$\tan \delta = \sin a \tan \omega = \cos (90^0 - a) \tan \omega, \text{ or } \tan \omega = \frac{1}{\sin a} \tan \delta,$$

we have

$$\delta = \omega - \frac{1 - \sin a}{1 + \sin a} . \sin 2\omega + \&c.$$

$$\omega = \delta + \frac{1 - \sin a}{1 + \sin a} . \sin 2\delta + \&c.$$

$$\text{But } \frac{1 - \sin a}{1 + \sin a} = \frac{1 - \cos (90 - a)}{1 + \cos (90 - a)} = \tan^2 \tfrac{1}{2} (90^0 - a)$$

$$= \tan^2 \tfrac{1}{2} u, \text{ (if } 90^0 - a = u);$$

$$\therefore \; \omega - \delta = \tan^2 \tfrac{1}{2} u . \sin 2\omega - \tfrac{1}{2} . \tan^4 \tfrac{1}{2} u . \sin 4\omega + \&c. \ldots (\gamma),$$

$$\omega - \delta = \tan^2 \tfrac{1}{2} u . \sin 2\delta + \tfrac{1}{2} . \tan^4 \tfrac{1}{2} u . \sin 4\delta + \&c. \ldots (\delta).$$

The series (γ), (δ) are called the *reductions of the declination to the solstitial declination*, in terms of the right ascension and obliquity, or of the right ascension and declination. By (δ) the obliquity is practically determined. If the Sun be near the solstice, this series is very convergent; and if δ be found on several successive noons at this time, the mean of the corresponding values of ω gives the obliquity with great exactness.

176. The reduction of the declination to the solstitial declination, in terms of the *longitude*, may be deduced from the series (γ), (δ), or more readily from the equation

$$\sin \delta = \sin \omega . \sin l.$$

Let $\delta = \omega - y$, $l = 90^0 - u$, where u is the Sun's distance in longitude from the solstice, and both y and u are very small;

$$\therefore \; \sin (\omega - y) = \sin \omega \cos u,$$

$$\text{or } \sin \omega \left\{1 - 2 (\sin \tfrac{1}{2} y)^2\right\} - \cos \omega \sin y = \sin \omega \left\{1 - 2 (\sin \tfrac{1}{2} u)^2\right\};$$

$$\therefore \; \sin y \, (1 + \tan \tfrac{1}{2} y \tan \omega) = 2 (\sin \tfrac{1}{2} u)^2 \tan \omega,$$

$$\text{or } y \, (1 + \tfrac{1}{2} y \tan \omega) = 2 (\sin \tfrac{1}{2} u)^2 \tan \omega;$$

therefore, for a first approximation, $y = 2\,(\sin\frac{1}{2}u)^2 \tan\omega$,

consequently, $y = 2\,(\sin\frac{1}{2}u)^2 \tan\omega\,(1 - \frac{1}{2}y\tan\omega)$,

$$= 2\left(\sin\frac{u}{2}\right)^2 . \tan\omega - 2\left(\sin\frac{u}{2}\right)^4 . (\tan\omega)^3 \quad (\epsilon).$$

In the equations α, β, γ, δ, ϵ the angles ω, l, y, &c. are expressed in circular measures; expressed in seconds, the last equation becomes

$$y = \frac{2\tan\omega}{\sin 1''} . \sin^2\tfrac{1}{2}u - \frac{2\tan^3\omega}{\sin 1''} . \sin^4\tfrac{1}{2}u + \ldots;$$

and the equations α, β, γ, δ may be reduced to seconds in the same way.

177. The formulæ γ, δ, ϵ, afford the most exact means of finding the obliquity. By Art. 169, a near value of ω may be obtained, from which, and an observed declination of the Sun near the solstice, u may be found by means of the equation

$$\sin\delta = \sin\omega . \sin l = \sin\omega . \cos u.$$

Substituting these values in the equation (ϵ), a value of y is obtained, and if, from n declinations, δ, δ', &c. near the solstices, n reductions y, y', &c. are deduced, the obliquity

$$= \frac{\delta + \delta' + \&\text{c.} + y + y' + \&\text{c.}}{n}, \text{ very nearly.}$$

The first term of the equation (ϵ) suffices for finding the reduction y, if the observations are made within 7 or 8 days of the Sun's being in the solstice; the observations may be extended through double that time, if the second term be made use of.

178. *To find the position of the ecliptic in the heavens at any time, with reference to the meridian and horizon of any place;* i. e. *for a given day and hour, to find the latitude and longitude of the pole of the horizon.*

Let ♈ E (fig. 15*) be the equator,
♈ O.....................ecliptic,
HR......................horizon,
P, K, Z, their respective poles.

Let the meridian PZ cut the ecliptic, equator, and horizon, in C, c and H respectively, and let KZ cut ♈ O in N, and HO in I; then C is called the *Culminating Point* of the ecliptic, or the *Mid Heaven*, O the *Ascending* point, and N the *Nonagesimal Degree*, being the point of the ecliptic which is at the greatest height above the horizon. Now the arc joining the poles of two great circles measures their inclination, and being produced, cuts the circles 90^0 from their point of intersection; therefore, ZK is the inclination of the ecliptic to the horizon, and $NO = 90^0$; for the same reason, OI, EH, ♈ T, ♈ t, all equal 90^0, T and t being the intersections of the equator and ecliptic with the solstitial colure.

Let s be the place of the Sun in the ecliptic, S his place referred to the equator, by a declination circle; then the right ascension of the mid heaven (A) = ♈ c = ♈ $S + Sc$ = the Sun's right ascension + the Sun's distance from the meridian measured along the equator = the Sun's right ascension + the hour angle from noon; therefore A is known, since the day and hour are given.

Let l = ♈ N be the longitude of the nonagesimal,

and $a = HO$ the south azimuth of the ascending point.

Then, in the triangle ZPK, we have given

PK the obliquity $= \omega$,

PZ the co-latitude $= c$,

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* In the figure, the arcs CO, HR, &c., are supposed to be on the eastern hemisphere, and C ♈, c ♈ on the western, so that OC, or from the right hand side of the page to the left, is the direction of the diurnal motion, or from east to west, and CO is the direction from west to east, or the order of the signs. Since the Sun (s) has just passed through an equinox to the north side of the equator, and also just passed the meridian; the figure represents the position of the great circles of the sphere in the afternoon of a day at the end of March.

and $\angle ZPK = 180^0 - ZPT = 180^0 - (♈ T - ♈ c) = 90^0 + A$, to find

$$\angle ZKP = \quad Nt \quad = ♈ t - ♈ N \ldots\ldots\ldots = 90^0 - l,$$

$$\angle PZK = CZN = HO - OI \ \ldots\ldots\ldots = a - 90^0,$$

and $ZK = NI$, the altitude of the nonagesimal;

therefore by Napier's analogies,

$$\tan \tfrac{1}{2}(a - l) = \frac{\cos \frac{1}{2}(PZ - PK)}{\cos \frac{1}{2}(PZ + PK)} \cdot \cot \tfrac{1}{2}(90^0 + A),$$

$$\tan \left\{90^0 - \tfrac{1}{2}(a + l)\right\} = \frac{\sin \frac{1}{2}(PZ - PK)}{\sin \frac{1}{2}(PZ + PK)} \cdot \cot \tfrac{1}{2}(90^0 + A),$$

$$\text{and } \sin ZK = \sin PK \cdot \frac{\sin ZPK}{\sin PZK} = \frac{\sin \omega \cdot \cos A}{- \cos a},$$

from which equations, a, l, ZK may be found.

The inclination of the ecliptic to the horizon, and the azimuth of the ascending point, are all that are required for ascertaining the position of the ecliptic with respect to the horizon; but l, the longitude of the nonagesimal, will be required in a subsequent Chapter.

179. When the first point of Aries rises, which it does in E, the solstitial colure and the meridian coincide because their poles coincide, and therefore K falls on the meridian; but since the Sun in moving eastward through Aries, passes into north declination, the part of the ecliptic situated to the east of Aries lies north of the equator; therefore, when Aries rises in north latitudes, $♈ m$ the part of the ecliptic which is west of Aries, falls below the equator $♈ n$, and cuts the meridian below c, and K lies in PZ or PZ produced, and not on the other side of P. As Aries ascends, or moves in the direction Ec, K moves in the same direction, and the angle ZPK increases; therefore, since the sides ZP, PK, are constant, ZK, or the inclination of the ecliptic and horizon, increases continually, from the time when K is on the meridian, or Aries rises, till $ZPK = 180^0$, when the solstitial colure again coincides with the meridian, and Aries sets.

The diurnal path of K is the polar circle; the greatest and least values of the inclination of the ecliptic to the horizon are, co-latitude + obliquity, and co-latitude $\sim$ obliquity, respectively.

180. *Given the right ascension and declination of a star to find its latitude and longitude; and conversely.*

Let $♈ D$, (fig. 16.) the right ascension of the star S, $= a$,
SD.............. declination....................... $= \delta$,
$♈ L$.............. longitude .... $= l$,
SL............... latitude .... $= \lambda$,
$\angle L ♈ D$.............. the obliquity $= \omega$, and $S ♈ D = \phi$.

Then, in the right-angled triangle

$$S ♈ D,\ \cot\phi = \sin a \,.\, \cot\delta \text{............} (1),$$

$$\cos\phi = \tan a \,.\, \cot ♈ S \text{.........} (2),$$

$$S ♈ L,\ \cos(\phi - \omega) = \tan l \,.\, \cot ♈ S \text{.........} (3),$$

$$S ♈ D,\ \sin\delta = \sin\phi \,.\, \sin ♈ S \text{.........} (4),$$

$$S ♈ L,\ \sin\lambda = \sin(\phi - \omega) \,.\, \sin ♈ S\ (5),$$

therefore, from (2) and (3),

$$\tan l = \frac{\cos(\phi - \omega) \,.\, \tan a}{\cos\phi},$$

and from (4) and (5), $\sin\lambda = \dfrac{\sin(\phi - \omega) \,.\, \sin\delta}{\sin\phi}$;

from which equations l and λ may be found, by means of the subsidiary angle ϕ, obtained from equation (1).

181. The right ascensions and declinations of the Sun and stars are deduced from observation by means of the sidereal clock and transit instrument, and the mural circle, as explained in Articles 110 and 115; the longitudes and latitudes not being subjects of immediate observation, are deduced by computation from right ascensions and declinations, and are useful

in the theory of the aberration of light, in certain methods founded on the occultations of stars by the Moon, and in the comparison of catalogues of stars made at different epochs; and afford the most direct method of finding the quantity of precession of the equinoxes.

182. The formulæ for finding the right ascension and declination from the latitude and longitude, may be deduced in exactly the same manner, by assuming $S \Upsilon L$ for the subsidiary angle ϕ; but it is obvious that they may be derived from the preceding, by changing λ into δ, l into a, and making ω negative, since the new plane to which the co-ordinates are referred, is measured downwards, and not upwards as in the preceding case. They are, therefore,

$$\cot\phi = \sin l \,.\, \cot\lambda,$$

$$\tan a = \frac{\cos(\phi + \omega) \,.\, \tan l}{\cos\phi},$$

$$\sin\delta = \frac{\sin\lambda \,.\, \sin(\phi + \omega)}{\sin\phi}.$$

183. Let OSL, the angle of position, $= S$,

$$OD = \theta; \quad \therefore\ SO = (\delta - \theta).$$

Then in the triangle ΥOD, $\tan\theta = \sin a \,.\, \tan\omega$ ...... (1),

.... $\sin\theta = \tan a \,.\, \cot \Upsilon OD$ .. (2),

.... SOL, $\cos(\delta - \theta) = \cot S \,.\, \cot \Upsilon OD$ ..(3);

therefore, from (2) and (3),

$$\cot S = \frac{\cos(\delta - \theta) \,.\, \tan a}{\sin\theta},$$

which determines S from the right ascension and declination, by means of the subsidiary angle θ.

184. Similiarly, if the arc of SL *produced*, between the ecliptic and equator, be taken for the subsidiary angle θ,

$$\tan\theta = \sin l \,.\, \tan\omega,$$

$$\cot S = \frac{\cos(\lambda + \theta) \,.\, \tan l}{\sin\theta},$$

which equations determine the angle of position, when the latitude and longitude of the star are given.

185. Produce DS, LS (fig. 17.) to meet the poles of the equator and ecliptic in P and K, and let KP meet the equator and ecliptic in Q and C; then KP is the solstitial colure, and ♈ is its pole, therefore the arcs ♈C, ♈Q, ♈P, ♈K, each $= 90^0$, and also the angles ♈KP, ♈PK, &c. are right angles;

$$\therefore\ SPK = 90^0 + a,\quad SKP = 90^0 - l,\quad PK = \omega,$$

$$SK = 90^0 - \lambda,\quad SP = 90^0 - \delta,$$

PSK, the angle of position of the star, $= S$.

$$\text{Hence } \cos SPK = -\sin a,\quad \sin SPK = \cos a,$$

$$\cos SKP = \quad \sin l,\quad \sin SPK = \cos l,$$

and applying these values and the fundamental formulæ of spherical Trigonometry, viz.

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

$$\cot A \sin C = \cot a \sin b - \cos C \cos b,$$

$$\sin a \sin B = \sin A \sin b,$$

(where a, b, c, A, B, C, represent the sides and opposite angles of any spherical triangle) to the triangle SPK, the following equations may be easily obtained:

$$\tan a = \cos \omega \,.\, \tan l - \tan \lambda \,.\, \sin \omega \operatorname{sect} l \ldots\ldots (1),$$

$$\sin \delta = \sin \omega \,.\, \cos \lambda \,.\, \sin l + \cos \omega \,.\, \sin \lambda \ \ldots\ (2),$$

$$\tan l = \cos \omega \,.\, \tan a + \sin \omega \,.\, \tan \delta \,.\, \operatorname{sect} a \ \ldots\ (3),$$

$$\sin \lambda = \cos \omega \,.\, \sin \delta - \sin \omega \,.\, \cos \delta \,.\, \sin a \ \ldots\ (4),$$

$$\cos \delta \,.\, \cos a = \cos \lambda \,.\, \cos l \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (5),$$

$$\cot S = \cot \omega \,.\, \cos \delta \,.\, \operatorname{sect} a + \tan a \,.\, \sin \delta \ \ldots\ (6),$$

$$= \cot \omega \,.\, \cos \lambda \,.\, \operatorname{sect} l - \tan l \,.\, \sin \lambda \ \ldots\ (7).$$

The figure represents the first quadrant of the sphere; in the second quadrant, SKP would $= l - 90^0$, and in the same way, that angle and SPK would alter in the other quadrants; but the algebraical results above deduced are of course general, and independent of the particular position of the geometrical figure whence they were derived.

186. We are now enabled to explain the method of making a catalogue of the fixed stars, and of registering their places either with respect to the ecliptic or the equator.

Having found (Art. 171.) the Sun's right ascension, let the hand of the sidereal clock be so placed, that the time by the clock, converted into degrees at the rate of 15° to one hour, may give the Sun's right ascension, reckoned from the true equinox, when on the meridian. Then, when Aries is on the meridian, the clock will point to 24^h, and if t^h be the time by the clock at any other instant, $15^0.t$ will be the right ascension of all fixed stars then on the meridian, which may therefore be found. The declinations of the stars may be deduced, as that of the Sun was (Art. 167.), by subtracting the observed meridian zenith distances from the latitude; considering zenith distances which are north of the zenith, negative, and all stars with negative declinations to be south of the equator. When the stars are thus registered according to their right ascensions and declinations, a catalogue, in which they are ranged according to their latitudes and longitudes, may be formed from it by Art. 180. On account of the small changes of place of the fixed stars alluded to in Art. 72 and 76, the right ascensions, declinations, &c. observed at one time, will be different from those observed at another; but since the law of these changes is known, we are able, from the observations made at different periods, to calculate the right ascensions and declinations such as they would have been found if all the observations had been made at the same given epoch. The use of this catalogue is for a variety of purposes already explained, such as to find the latitude, to regulate the clock, to verify the adjustment of instruments, &c.; as well as to ascertain whether any new bodies may hereafter visit our system, or any of the known ones depart from it.

CHAPTER VI.

ON TIME.

187. ASTRONOMERS, as has been said, distinguish three kinds of days, the *Sidereal*, the *Solar*, and the *Mean Solar*, each of which is divided into 24 equal parts or hours, each hour into 60 minutes, and each minute into 60 seconds.

The sidereal day is the interval between two successive transits of the true first point of Aries (♈), and begins when that point is on the meridian. The sidereal day may be considered the same as the time of the Earth's rotation*, or as the time between two successive transits of the same fixed star over the meridian.

188. The sidereal time in common use among *Astronomers* expresses the actual hour angle from the meridian, westwards, of the true equinoctial point. The *sidereal time at mean noon* given in the Nautical Almanac, is the angular distance of the true vernal equinox, from the meridian, at the instant of mean noon, and is calculated by the formula,

$$\text{Sid. } T = \frac{1}{15}(\odot\text{'s mean long.} - 15'',395 \sin ☽\text{'s ☊} - 1'',225 \sin 2 \odot\text{'s true long.})$$

It is therefore affected by the equation of the equinoxes; and is not, strictly speaking, a *mean* or uniformly increasing quantity. It ought, therefore, to be termed *apparent sidereal time*, in the same manner as apparent solar time reckons from the actual arrival of the Sun's center on the meridian. And in like manner, as mean solar time is reckoned from the arrival of an imaginary Sun, moving uniformly with its mean velocity, so *mean sidereal* time (whose expression would be simply $\frac{1}{15}$ ⊙'s mean long.) would be reckoned from the transit of (not the *true*, but) the *mean* equinoctial point. The smallness of the fluctuations to which a clock, regulated to *apparent* sidereal time compared with one regulated to mean sidereal time, is subject, being at the utmost only $2^s,3$ in a period of 19 years, has prevented the practical inconvenience of this from being felt; no clock being sufficiently perfect to go during so long a period without frequent re-adjusting.

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* If ♈ were a fixed point, the sidereal day would be *equal* to the interval between two transits of the same fixed star, or to the time of the Earth's rotation; but the axis of the Earth, as stated in Art. 77, and as will be noticed more fully in the Chapter on Precession, describes a conical surface round the axis of the ecliptic in 25680 years, in consequence of which, the intersection of the ecliptic and equator moves along the ecliptic through 50″,2, yearly, in a direction contrary to the order of the signs. This motion causes the length of the sidereal day, as defined in the text, to differ from the time of the Earth's rotation, and from the interval between two successive transits of the same fixed star. There are also other causes (Chap. x, xi), which alter slightly the real or apparent right ascensions of the fixed stars, and therefore render unequal the intervals between the successive transits of different stars, or of the same star at different times. The difference, however, of a sidereal day, and of the interval between two successive transits of the same fixed star, except for stars very near the poles, is too small to be noted by the clock, and can only be detected in its accumulation for a long period.

189. A *Solar day* is the interval between two successive transits of the Sun's center over the same meridian, and begins at noon. This is sometimes called the *true* day, and sometimes the *apparent* day, because marked by a real phenomenon, viz. the appearance of the Sun on the meridian, and not being, as the sidereal and mean solar days are, the result of calculation. The *Mean Solar* day, as its name imports, is the mean of solar days, and is determined by the transits of an imaginary Sun, which describes the equinoxial with the Sun's mean motion in right ascension or longitude.

The difference between the apparent and mean time is called the *equation of time*, by the addition of which to apparent time, the mean time may be found. The means of determining the equation of time will be given in a subsequent Chapter; here it may be taken for granted, that when either the apparent or mean time is known, the other can be found, if the Sun's place in the ecliptic be given.

190. The astronomical day begins at mean noon, and is continued through the 24 hours to the following mean noon; the civil day commences at the preceding midnight and is counted only to 12^h or noon, when the 12 hours are reckoned over again to the next midnight; the civil reckoning therefore is always 12^h in advance of the astronomical reckoning. Thus

Jan. $1^d.\,7^h.\,49^m$, Jan. $1^d.\,15^h.\,35^m$, astronomical time, correspond respectively to Jan. $1^d.\,7^h.\,49^m$, p.m., Jan. $2^d.\,3^h.\,35^m$, a.m. civil time.

191. As the Earth revolves uniformly, a given meridian separates itself from the mean Sun by an angle of 360^0 in 24 mean solar hours; therefore, if h^0 be the distance of the mean Sun from the meridian, or the hour angle, $\frac{h}{15}$ will be the mean solar time in hours. In the same manner, if h^0 be the hour angle of the true Sun, or of ♈, $\frac{h}{15}$ will be the true, or sidereal time, respectively.

Since any phenomenon, which is visible at the same instant of absolute time, will appear to happen an hour later for every 15^0 of east longitude, and an hour earlier for every 15^0 of west longitude; if the time of any observation be given, the absolute instant of its happening is not determined, unless the meridian from which the time is reckoned be given.

The time used in this work is always mean time, unless the contrary be expressed, and is that to which clocks and watches are usually adapted. Astronomical clocks are often regulated to sidereal time. The hours pointed out by sun dials are in solar time.

192. Mean solar time, according to another mode of computation, is sometimes called *equinoctial time*, which is the mean time elapsed since the instant of the preceding mean vernal equinox. Thus in 1838 the instant of the mean vernal equinox was March $22^d.463822$, consequently at mean noon of March 23, 24, &c. the equinoctial time was $0^d.536178$, $1^d.536178$, &c.; and so on till the instant of the mean vernal equinox of 1839. The advantage of this mode of reckoning mean time is that it saves the necessity, when speaking of the time of an event's happening, of mentioning the place where it was observed or computed. Thus it is the same thing to say that a comet passed its perihelion on March 24, 1838, at $5^h.\,56^m.\,21^s,5$ mean time at Paris, or at $1838^y.\,1^d.\,18^h.\,39^m.\,5^s,7$ equinoctial time; but the former date depends on the locality of Paris, whereas the latter expresses the interval elapsed since an epoch

common to all the world, and identifiable independently of all localities.

To convert mean solar into equinoctial time, we must add to the Greenwich mean time the equinoctial time at mean noon of the same day at Greenwich. Thus in the above example, Paris being $9^{m}.21^{s},5$ east of Greenwich, subtracting this from the Paris time we get $5^{h}.47^{m}$, for the Greenwich time, to which adding $1^{d}.536178$, or $1^{d}.\,12^{h}.\,52^{m}.\,5^{s},7$, the equinoctial time at Greenwich mean noon of March 24, we get $1^{d}.\,18^{h}.\,39^{m}.5^{s},7$ the mean equinoctial time from the vernal equinox of 1838.

193. *To compare the lengths of a sidereal and mean solar day.*

The mean Sun in the equator, which moves with the Sun's mean motion in longitude, describes an angle of $59'.8'',33$ in a mean solar day (Art. 60.); therefore in the interval between two successive transits over the same meridian, the Earth must revolve through $360^{0}.59'.8'',33$; but in a sidereal day the Earth revolves through 360^{0};

$$\therefore \frac{\text{a sidereal day}}{\text{a mean solar day}} = \frac{360^{0}}{360^{0}.59'.8'',33}.$$

194. *To convert any duration of sidereal into the corresponding duration of mean solar time, and conversely.*

Since the number of hours, minutes, or seconds, in any given interval of absolute time must be inversely as the length of one of them; therefore, if m be any time expressed in mean hours, minutes and seconds, s the same time expressed in sidereal hours, minutes and seconds,

$$\therefore\ m = s\,.\frac{360^{0}}{360^{0}.59'.8'',33} = s - \frac{59'.8'',33}{360^{0}.59'.8'',33}\,.\,s \ldots\ (1),$$

$$s = m\,.\frac{360^{0}.59'.8'',33}{360^{0}} = m + \frac{59'.8'',33}{360^{0}}\,.\,m \ldots\ldots\ (2),$$

By equation (1) any period, expressed in sidereal time, may be reduced to mean, and the converse by equation (2).

195. These reductions may be facilitated by calculating the quantities to be added or subtracted for one day, one hour,

and one minute. To reduce any time expressed in sidereal hours, minutes and seconds, to mean hours, &c.

for each day subtract $\frac{59'.8'',33}{360^0.59'.8'',33}$ days $= 3^m.55^s,9093$,

.... hour .... $\frac{3^m.55^s,9093}{24}$ ...... $=$ $9^s,8296$,

.... minute ...... $\frac{9^s,8296}{60}$ ...... $=$ $0^s,1638$,

.... second .... $0^s,00273$.

Similarly, any period, expressed in mean time, may be reduced to sidereal, by adding $\frac{59'.8'',33}{360^0}$. days, or $3^m.56^s,5553$ for each day, $9^s,8568$ for each hour, $0^s,1643$ for each minute, and $0^s,00273$ for each second.

Thus the sidereal interval $10^h.8^m.40^s = 10^h.7^m.0^s,1$ mean time; and the mean interval $2^h.29^m.6^s,18 = 2^h.29^m.30^s,67$ sidereal time.

A sidereal day $= 24^h - (3^m.55^s,9093) = 23^h.56^m.4^s$ mean time; and a mean solar day $= 24^h.3^m.56^s$ sidereal time: a clock which marks either, may be regulated to the other by a proper adjustment of the pendulum.

196. *Given the sidereal time of any phenomenon, under a given meridian, to find the corresponding mean solar time, and conversely.*

Let PZA (fig. 18.) be the given meridian, and at the instant the phenomenon happens, the sidereal time of which is s and the mean time m, let ΥMA be the position of the equator, Υ being the first point of Aries and M the mean Sun. Also, let n be the place the mean Sun occupied when it last crossed the meridian, that is, at the preceding mean noon; then since $59'.8'',33$ is the increase of the mean Sun's right ascension in 24 mean solar hours, nM, the increase since it left the meridian (that is in m mean solar hours)

$$= \frac{m}{24}(59'.8'',33).$$

Now $\Upsilon A = \Upsilon n + nM + MA = \Upsilon n + \frac{m}{24}(59'.\,8'',33) + MA$;

$$\therefore\ \frac{\Upsilon A}{15} = \frac{\Upsilon n}{15} + \frac{m}{24} \cdot \frac{59'.\,8'',33}{15} + \frac{MA}{15}.$$

Let $\frac{59'.\,8'',33}{15} = 3^{m}.\,56^{s},55$ be denoted by b, and the mean Sun's right ascension at the preceding mean noon expressed in time, or $\frac{\Upsilon n}{15}$, by a; $\therefore$ (Art. 191) $s = a + m + \frac{b}{24} m$,

$$\text{and } m = (s - a) \cdot \frac{24}{b + 24} = s - a - (s - a) \cdot \frac{b}{b + 24}.$$

Thus, if the sidereal time of any phenomenon be $5^{h}.\,47^{m}.\,56^{s}$, the corresponding mean time, supposing the mean Sun's right ascension when on the meridian $= 1^{h}.\,17^{m}.\,6^{s}$, will be $4^{h}.\,30^{m}.\,5^{s},7$.

197. To find the mean time of a star's transit.

Since the star is then on the meridian, $s = \alpha$, where α denotes the star's right ascension in time;

$$\therefore\ m = \alpha - a - (\alpha - a) \cdot \frac{b}{24 + b},$$

which is the mean time of the star's transit.

198. To find the mean time of a planet's transit.

Let A (fig. 18.) be the planet on the meridian, and m the corresponding mean time; P the right ascension at the preceding mean noon, and e the increment of right ascension in 24 mean hours, both expressed in time.

$$\text{Then } \frac{A\Upsilon}{15} = P + \text{increment of } P \text{ in the time } m,$$

$$= P + e \cdot \frac{m}{24},$$

$$\text{and } \frac{M\Upsilon}{15} = a + b \cdot \frac{m}{24}; \text{ (Art. 196.)}$$

therefore, subtracting,

$$\frac{AM}{15} \text{ or } m = P - a + (e - b) \cdot \frac{m}{24};$$

$$\therefore\ m = \frac{(P - a) \cdot 24}{24 - e + b}$$

$$= P - a + \frac{e - b}{24 - e + b} \cdot (P - a).$$

If the body's motion is retrograde, e is negative, Art. 103.

199. *To find the hour angle described by a planet in a given interval of mean solar time.*

Let b^0 (59′. 8″,33) be the mean Sun's motion in Æ in 24 hours,

n^0 .... planet's ....

Then, if the planet and Sun be together on the same meridian, when the Sun is next on the meridian, the planet will be at a distance $n^0 - b^0$, or, instead of having separated from the meridian by 360^0, will have separated only by $360^0 - n^0 + b^0$, in 24 mean solar hours; therefore, if T be the time corresponding to an hour angle P,

$$P = \frac{T}{24} \cdot (360^0 - n^0 + b^0).$$

Hence $T = \dfrac{P}{360^0 - n^0 + b^0} \cdot 24^{\mathrm{h}}$, in mean solar hours; and the time between two successive transits of a planet

$$= \frac{360^0}{360^0 - n^0 + b^0} \cdot 24^{\mathrm{h}}.$$

To determine the mean time by observation.

200. First method. *By observing the time of transit of the Sun, or a known star.*

If the Sun be the body observed, the time of transit will be apparent noon, and the equation of time being added, will give the corresponding mean time. If a star be the body observed, its right ascension converted into time, will be the sidereal time, from whence the mean time may be found.

201. Second method. *By an observed altitude of the Sun.*

Let z be the Sun's zenith distance = 90° − the observed altitude, Δ his north polar distance, h the hour angle, and c the colatitude; then in the triangle PZS (fig. 10.)

$$\sin \tfrac{1}{2} h = \sqrt{\frac{\sin \frac{1}{2}(z + \Delta - c) \cdot \sin \frac{1}{2}(z + c - \Delta)}{\sin c \cdot \sin \Delta}};$$

whence h, and therefore $\frac{h}{15}$, the apparent time, may be found.

If z be the zenith distance, and Δ the north polar distance of a star, whose right ascension is a, $\frac{a \pm h}{15}$ is the sidereal time; + or − according as the star is west or east of the meridian; for if D be the star's place referred to the equator (fig. 18.), $\Upsilon A = \Upsilon D - AD = a - h$; similarly if the star is west of the meridian.

202. Given a small error in the observed altitude, to find the corresponding error in the time computed from it.

Let z, h, be the true zenith distance and hour angle, z', h', the erroneous observed zenith distance and computed hour angle, Δ the north polar distance and c the co-latitude;

$$\therefore \cos h - \cos h' = \frac{\cos z - \cos z'}{\sin c \sin \Delta},$$

$$\text{or } 2 \sin \frac{h' - h}{2} \sin \frac{h + h'}{2} = \frac{1}{\sin c \sin \Delta} 2 \sin \frac{z' - z}{2} \sin \frac{z' + z}{2},$$

or, since $z' - z$ and $h' - h$ are very small,

$$(h' - h) \sin h = \frac{\sin z}{\sin c \sin \Delta}(z' - z),$$

$$\text{but } \frac{\sin A}{\sin h} = \frac{\sin \Delta}{\sin z} \text{ (}A\text{ being the azimuth)};$$

$$\therefore\ h' - h = \frac{z' - z}{\sin c \sin A};$$

$$\therefore \text{ the error in time} = \frac{1}{15^0} \cdot \frac{1}{\sin c \sin A} \cdot \text{error in altitude.}$$

Hence, the error in time, from a given error in altitude, is less the greater $\sin A$ is, or the nearer the body is to the prime vertical, at the instant of observation.

203. Third method. *By equal or corresponding altitudes of the Sun or a known star, taken before and after culmination.*

Let the star be observed at equal altitudes on each side of the meridian, then, since the declination is invariable, the corresponding hour angles are equal, and the time by the clock, of the star's transit = half the sum of the times of observation; if this time be compared with the known sidereal time of the star's transit (Art. 186), the deviation of the time marked by the clock from sidereal time, at the instant of culmination, is obtained.

The time, by the clock, of the star's culmination, will be less liable to be vitiated by errors of observation, if several pairs of observations are made, at equal altitudes, on each side of the meridian, and if the sum of all the times, divided by twice the number of observations, be taken for the time of the star's transit.

204. If the deviation of the clock from sidereal time be found after one or more days, the rate, or daily gain of the clock over sidereal time is ascertained. If the clock gain e'' in 24 sidereal hours, in t^{h} the gain is $\frac{t}{24} \cdot e''$, and if τ^{h} be the interval by the clock, corresponding to an interval of t sidereal hours,

$$t = \tau - \frac{t}{24} \cdot e'';$$

$$\therefore\ t = \frac{24^{\text{h}}}{24^{\text{h}} + e''} \cdot \tau = \frac{86400}{86400 + e} \cdot \tau.$$

If the clock loses with respect to sidereal time, e is negative.

Hence, if h^0 be the hour angle described in τ^{h} by the clock,

$$h^0 = 15^0 . t = 15^0 . \frac{86400}{86400 + e} . \tau,$$

$$\text{and } \tau = \frac{86400 + e}{86400} . \frac{h}{15} .$$

In exactly the same manner, as in the last two Articles, the deviation of a clock from *mean* or *apparent* time may be ascertained, and the *mean* or apparent interval corresponding to any interval by the clock.

205. When equal altitudes of the Sun are taken before and after noon, the time of transit is not exactly the semi-sum of the times of observation, on account of the Sun's change of declination. To correct for this, let h and h' be the hour angles, δ and δ' the declinations of the Sun, corresponding to the common zenith distance z;

$$\therefore \cos z = \cos l . \cos \delta . \cos h + \sin l . \sin \delta.$$

Then, since z and l remain the same whilst h and δ change, δ is a function of h, and the equation for determining $d_h \delta$ is, $0 = (\sin l \cos \delta - \cos l \cos h \sin \delta)\, d_h \delta - \cos l . \cos \delta . \sin h$; but $d_h \delta =$ limiting ratio of contemporary increments $= \dfrac{\delta' - \delta}{h' - h}$ nearly;

$$\therefore h' - h = \frac{\tan l - \cos h . \tan \delta}{\sin h} . (\delta' - \delta);$$

$$\therefore \tfrac{1}{2} . \frac{h' - h}{15^0} = \frac{\tan l - \cos h \tan \delta}{\sin h} . \frac{\delta' - \delta}{30^0};$$

and subtracting this quantity, which is expressed in parts of an hour, from the semi-sum of the times of observation, the time $\dfrac{h}{15^0}$ by the clock of the Sun's transit is obtained, supposing the clock to be adjusted to apparent time.

If the interval, by the clock, between the preceding and succeeding noons, is $24^{\text{h}} + e''$, and not 24^{h}, the correction is

$$\frac{1}{2}\cdot\frac{86400+e}{86400}\cdot\frac{h'-h}{15}$$
$$=\frac{1}{2}\cdot\frac{86400+e}{86400}\cdot\frac{\tan l-\cos h\,.\tan\delta}{\sin h}\cdot\frac{\delta'-\delta}{15},$$

which must be subtracted from the semi-sum of the times of observation, as before.

206. Fourth method. *From two altitudes of the Sun, or of a known star, and the time between.*

This method, which is adopted only when the latitude is unknown, has been already investigated in Art. 156, where the latitude and hour angle were found from two altitudes, and the time between; having found the hour angle, the apparent time is known, if the Sun be the body observed, and the sidereal time, if the body observed be a known star.

The four methods which have been described are those most frequently adopted in finding the time by observation; the three last may be applied generally by sea or land, but the first, in which a transit instrument is required, can only be made use of in an Observatory. There is a great variety of other ways in which the time may be found from observations; and as the investigation in general is attended with no difficulty, it will be sufficient here to give one of them which can be most frequently applied, as it requires only a few simple contrivances which can always be procured.

207. *To find the time when two known stars are on the same vertical.*

Let Z (fig. 20.) be the zenith, P the pole, S and S' the two stars in the vertical ZS; draw Pp perpendicular to ZS, and let δ and δ' be the declinations of the stars, a and a' their right ascensions, and h the hour angle ZPS; then $SPS'=a'-a$, and if $SPp=\phi$, $S'Pp=a'-a-\phi$; therefore, applying Napier's rules to the triangles SPp, $S'Pp$,

$$\cos\phi=\tan\delta\,.\tan Pp\ldots\ldots\ldots\ldots(1),$$

$$\cos(a'-a-\phi)=\tan\delta'\,.\tan Pp\ldots\ldots\ldots\ldots(2);$$

$$\therefore\ \frac{\cos(a'-a-\phi)-\cos\phi}{\cos(a'-a-\phi)+\cos\phi}=\frac{\tan\delta'-\tan\delta}{\tan\delta'+\tan\delta}:$$

$$\text{Or,}\quad \tan\tfrac{1}{2}(a'-a)\,.\,\tan\tfrac{1}{2}(2\phi-a'+a)=\frac{\sin(\delta'-\delta)}{\sin(\delta+\delta')};$$

$$\therefore\ \tan\tfrac{1}{2}(2\phi-a'+a)=\frac{\sin(\delta'-\delta)}{\sin(\delta'+\delta)}\,.\cot\tfrac{1}{2}(a'-a),$$

which determines ϕ.

Again, in the right-angled triangle ZPp,

$$\cos(h+\phi)=\tan l\,.\tan Pp$$
$$=\tan l\,.\cos\phi\,.\cot\delta \text{ from } (1),$$

from which h and therefore the time of the observation may be found.

The instant when the stars are in the same vertical may be easily ascertained by suspending a plumb-line, and observing when they both cross the line at the same time. The pole star ought to be one of the stars chosen for this observation, because, on account of the small north polar distance, a star which is in the same vertical with the pole star must be near the meridian, and, as its motion is then nearly horizontal, the time when it crosses the vertical line can be noted more exactly.

208. If A be the azimuth of the vertical plane in which the stars are together,

$$\sin l=\cot A\,.\cot(h+\phi);$$

$$\therefore\ \cot A=\sin l\,.\tan(h+\phi).$$

If S and S' be two positions of the *same* circumpolar star, observed after an interval of $2t$ sidereal hours; $SPS'=15^0.\,2t$; $\therefore\ \phi=15^0.\,t$, and the equation for determining ϕ may be dispensed with.

209. By whatever method the time, whether mean or sidereal, is found, if it be compared with that shewn by the clock, the quantity by which the clock is fast or slow with respect to mean or sidereal time, may be ascertained, and also the daily gain, as in Art. 204.

The rate of a common watch may be easily found by observing with it the instant of disappearance of a bright star behind a building, or any other fixed object. This takes place always at the same point, after an interval of $23^h.\,56'.\,4''$, or $3'.\,56''$ earlier every night, by a watch regulated to mean time; therefore, by observing on different nights the times of disappearing of several bright stars, the watch's rate may be ascertained and corrected. This method is available where there is no telescope, the only precaution required being for the eye of the observer to be placed as near as possible in the same place, and to select stars which have small declinations, in order that the time of their disappearance may be noted more readily.

Calculation of the lengths of the various kinds of Years.

210. *To find the length of the tropical year.*

If the Sun be on any meridian when in ♈ at the beginning of the year, after 365 transits he will not have reached ♈, and after 366 will have passed it; therefore the tropical year consists of something more than 365^d. The length may be found most accurately by comparing observations made at distant periods. The Sun's right ascension can be found from his observed declination, and by comparing the right ascension when the Sun was near ♈ with observations made 36525 days, or nearly 100 years before, Delambre found the Sun's motion in right ascension in that time to be $100 \times 360^0 + 45'.\,45''$. In one year the Sun separates from ♈ by 360^0;

$$\therefore \text{ the length of a tropical year} = 36525^d.\,\frac{360^0}{100 \times 360^0 + 45'.\,45''}$$

$$= 365^d.\,5^h.\,48'.\,51'',6.$$

The year thus found is the mean tropical year, as, on account of irregularities in the motion of the Sun and of ♈, the tropical years differ a little from each other. The tropical year at any time may be obtained, (by finding A^0, the increase of the Sun's right ascension in 365 days,) from the equation,

$$\text{length of the tropical year} = 365^d.\,\frac{360}{A}.$$

211. The correct value of the mean tropical year is 365.242218 mean solar days. This length is not convenient for

civil purposes, for which it is requisite that the year should consist of an exact number of days; to insure which, and at the same time to make the Sun, on whose declination the seasons depend, be near the vernal equinox at the commencement of the civil year, Julius Cæsar made every 4th year to consist of 366^d, and other years of 365^d, (See Art. 82). This supposes 4 tropical years to contain $4 \times 365 + 1$ days, or one year to consist of $365^d.25$, which exceeds the true value by $0^d.007782$, so that in 1000 years the civil year would begin more than 7^d after the Sun had passed the equinox. To correct this accumulating error, Pope Gregory made common years to consist of 365^d, and intercalated one day every fourth year as before, but omitted the intercalation in those years completing centuries, when the number of centuries is not divisible by 4, or only intercalated 97^d in 400^y instead of 100^d, as in the Julian correction. This supposes the length of one year to be $365^d + \frac{97}{400} = 365.2425$, which is too great by .000282, but the error scarcely exceeds a day in 4000^y.

If the Gregorian intercalation of 97^d in 400^y, or 970^d in 4000^y be omitted every 4000th year, the civil year would be $365^d + \frac{969}{4000} = 365.242250$, and it would require more than 30000 years to make the beginning of the civil and tropical year be a day apart.

The Gregorian intercalation is adopted in most Christian countries, and is made on the 29th of February, which has 28^d in common years, and 29^d in leap year.

212. The length of the sidereal year, by the most recent determination, is $365^d.6^h.9^m.10^s,7496$ or $365^d,256374$.

Since Aries moves with a slow annual motion of $50'',23$ to meet the Sun, the sidereal is greater than the tropical year, by the time the Sun takes to move over this space, and they are connected by the relation;

$$\therefore \frac{\text{a sidereal year}}{\text{mean tropical year}} = \frac{360^0}{360^0 - 50'',23},$$

this makes the mean tropical year, as stated above,

$$= 365^{d},242218 = 365^{d}.\ 5^{h}.\ 48^{m}.\ 47^{s},6352.$$

213. *To find the length of an anomalistic year.*

The apogee is not fixed, but moves through 11″,25 in the direction of the Sun's motion in the course of a year; therefore, the anomalistic year is longer than the sidereal, by the time which the Sun takes to move over this space;

$$\therefore \text{ the anomalistic year} = 365^{d},256374 \times \frac{360^{0} + 11'',25}{360^{0}};$$

$$\therefore \text{ the anomalistic year} = 365^{d},259544 = 365^{d}.\ 6^{h}.\ 13^{m}.\ 44^{s},6.$$

CHAPTER VII.

ON THE TIMES OF RISING AND SETTING OF THE HEAVENLY BODIES.

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214. *To find the length of the day at any place.*

Let Z (fig. 10.) be the zenith, P the pole, S the Sun at rising; then, $ZS = 90^0$, and ZPS is the hour angle from sunrise to mid-day. It is evident that the spherical triangle formed by P, Z and S, at sun-set, on the other side of the meridian, will be the same as PZS, if PS be constant; therefore, if $2t$ be the length of the day in solar hours, $ZPS = 15^0.t$.

Let l be the latitude $= 90^0 - PZ$,

δ the Sun's declination $= 90^0 - PS$;

$$\therefore \cos 15^0.t = \frac{\cos 90^0 - \sin l \,.\, \sin \delta}{\cos l \,.\, \cos \delta} = -\tan l \,.\, \tan \delta.$$

Hence also, if 2τ be the duration of the night in solar hours, since $2\tau = 24 - 2t$, or $t = 12 - \tau$,

$$\cos 15^0 . (12 - \tau) = -\tan l \,.\, \tan \delta,$$

$$\text{or } \cos 15^0 . \tau = \quad \tan l \,.\, \tan \delta.$$

215. If PS be not constant, the hour angles for the morning and evening are unequal, and therefore the times; to find the difference, let t be the length of the morning and t' of the evening, δ the Sun's declination at sun-rise, and $(\delta + \Delta)$ at sun-set, where Δ is very small, then (fig. 10. bis), if PS be the north polar distance at rising and PS' at setting, and we draw $S'B$ perpendicular to SP, SB is the increase of declination $= \Delta$, and $S'PS$ is the difference of the hour angles at sun-rise and sun-set $= 15^0\ (t' - t)$.

$$\text{But } S'PS = \frac{S'B}{\cos\delta} = \frac{SB \tan PSH}{\cos\delta} = \frac{\Delta}{\cos\delta}\,\frac{\sin l \sec\delta}{\sqrt{1-(\sin l \sec\delta)^2}},$$

because $\sin PSH = \sin l \sec\delta$;

$$\therefore\ t' - t = \frac{\Delta}{15^0}\,\frac{\sin l \sec\delta}{\sqrt{(\cos\delta)^2-(\sin l)^2}} = \frac{\Delta}{15^0}\cdot\frac{\sin l \sec\delta}{\sqrt{\cos(l-\delta)\cos(l+\delta)}}.$$

Hence, if Δ be positive, or the declination increases, the evenings are longer than the mornings, if Δ be negative, they are shorter; and at the solstices, when $\Delta = 0$, the mornings and evenings are equal, and then only noon is the middle of the day.

216. *To compare the lengths of the day at different times of the year.*

Suppose the observer in north latitude, and $2t$ as before the length of the day; therefore,

$$\cos 15^0.t = -\tan l.\tan\delta.$$

When $\delta = 0$, $\cos 15^0.t = 0$; $\therefore\ 2t = 12^h$, or the days and the nights are equal in all latitudes; as δ increases $15^0.t$ increases, till the Sun reaches his greatest north declination $23^0.28'$ at the time of the summer solstice, when the equation

$$\cos 15^0.t = -\tan l.\tan 23^0.28'$$

determines the duration of the longest day. As δ decreases from $23^0.28'$, t diminishes, and when the Sun again crosses the equator in Libra, at which time $\delta = 0$, the days are again 12^h long all over the world.

After passing the equator δ is negative, and the equation becomes

$$\cos 15^0.t = \tan l.\tan\delta;$$

therefore as δ increases t diminishes, and when $\delta = 23^0.28'$, the equation

$$\cos 15^0.t = \tan l.\tan 23^0.28',$$

determines the length of the shortest day. As the Sun moves from the winter solstice to the vernal equinox, δ diminishes from $23^0.28'$ to 0 and the days increase in length, till at the vernal equinox the day again is 12^h long.

If the Sun's declination = the co-latitude of the place of observation,

$$\cos 15\,t = -\tan l \,.\, \tan \delta = -1\,;$$

therefore, the day is 24^{h} long, and the Sun does not set during an entire revolution.

Hence in latitude $66^{0}.32'$ the Sun does not set for 24^{h} when at his greatest north declination $23^{0}.28'$; and in the higher latitudes the Sun will not set from the time when his declination = the co-latitude, till it again becomes equal to the same quantity after passing through its maximum $23^{0}.28'$.

In the same way it may be shewn that the Sun does not rise in latitude $66^{0}.32'$ for 24^{h} at the time of the winter solstice, and continues beneath the horizon when south of the equator, at all higher latitudes for the same time that he continued above the horizon when in north declination.

The preceding observations, applied to places in the southern hemisphere, shew that the length of the day there increases with the Sun's southern, and diminishes with his northern, declination, in such a manner that the length of the day at any time, at a place in north latitude, = the length of the night in the same latitude south. All this agrees with what was said in articles 52 and 53 of the Introduction.

217. It is only at the solstices, that the zenith distance of the Sun's center is the least when it is on the meridian; at other times, when, for instance, his north polar distance is increasing, the zenith distance of his center will be a minimum before the transit; for during a short interval before the transit it will be more increased from that cause, than diminished by the approach of the meridian to the Sun; similarly, when the Sun's north polar distance is diminishing, the zenith distance of his center will be a minimum after the transit.

To find the hour angle when the zenith distance of the Sun's center is minimum.

Suppose the Sun's north polar distance to be increasing, and let S (fig. 10.) be the position of the Sun's center at some instant before the transit when ZS (z) is a minimum.

Let D = the meridian zenith distance of the Sun's center,

a = the increase of the Sun's N.P.D. between two successive transits, and h = the hour angle ZPS,

$$\text{then } D = PS + \frac{a}{2\pi} . h = \Delta + mh, \text{ suppose};$$

$$\text{now } \cos z = \cos c \cos \Delta + \sin c \sin \Delta \cos h;$$

$$\therefore\ -\sin z\, d_h z = -\cos c \sin \Delta\, d_h \Delta + \sin c \cos \Delta \cos h\, d_h \Delta$$
$$- \sin c \sin \Delta \sin h,$$

$$\text{but } d_h z = 0,\ d_h \Delta = -m;$$

$$\therefore\ \sin h = m\,(-\cot c + \cot \Delta \cos h),$$

from which equation h may be determined. The same process evidently serves to determine the hour angle when a Planet's zenith distance is least.

Problems depending upon azimuths, altitudes, and hour angles.

218. A great variety of problems relating to azimuths, altitudes, &c. may be solved by means of the triangle PZS, (fig. 10.), in which P is the pole of the equator, Z the zenith, and S a heavenly body; if in this triangle we call

ZP the co-latitude $= c$, or $90^0 - l$,

PS ... co-declination of $S = \Delta$, or $90^0 - \delta$,

ZS ... zenith distance of $S = z$, or $90^0 - a$,

ZPS ... hour angle $= h$, and PZS the azimuth $= A$,

then any three of the quantities l, δ, z, h, A, being given, the other two may be found by the common rules of Trigonometry.

Ex. 1. *To find the azimuth of a heavenly body from its given declination and zenith distance at a given place.*

Here we have PZ, PS, ZS given, and PZS required;

$$\text{therefore } \cos A = \frac{\sin \delta - \sin l \cos z}{\cos l \sin z},$$

or, in a form adapted to logarithmic computation,

$$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}(\Delta + c - z)\sin\frac{1}{2}(\Delta + z - c)}{\sin z \sin c}}$$

Ex. 2. *To find the Sun's azimuth at rising.*

Here $z = 90^0$; therefore $\cos A = \dfrac{\sin\delta}{\cos l}$.

Ex. 3. *To find the Sun's azimuth, when his declination, and the time of day are given.*

Here *PZ*, *PS*, *ZPS* are given to find *PZS*; let ϕ be a subsidiary angle determined by the equation

$$\tan\phi = \cos h \,.\, \cot\delta;$$

therefore, since $\cot A = \dfrac{\tan\delta\cos l - \cos h\sin l}{\sin h}$,

$$\cot A = \cot h\,(\cot\phi\cos l - \sin l) = \frac{\cot h\cos(\phi + l)}{\sin\phi}.$$

If the azimuth, determined by any of the three last articles, be compared with that pointed out by the compass, we obtain the *variation of the compass*, or the deflection of the magnetic meridian from the true meridian.

Ex. 4. *To find when the Sun is due east on a given day.*

Here *ZS* is the prime vertical, and *PZS* is a right angle;

$$\text{therefore } \cos h = \cot l \,.\, \tan\delta.$$

If z be the corresponding zenith distance,

$$\sin\delta = \sin l \,.\, \cos z; \quad \therefore\ \cos z = \sin\delta \,.\, \operatorname{cosec} l.$$

Ex. 5. *To find the Sun's zenith distance at six o'clock.*

Here $ZPS = 90^0$, therefore $\cos z = \sin l \,.\, \sin\delta$.

Ex. 6. *To find the hour angle when a Star's motion is vertical, the declination of the Star being greater than the latitude of the observer.*

If a star S (fig. 22.) be nearer to the pole than the observer's zenith Z, its circle of diurnal motion will pass between Z and P; therefore PZS the azimuth, will increase till the vertical circle ZS touches the parallel Ss described by the star, and afterwards diminish. There will in the same manner be another maximum azimuth on the other side of the meridian, and at each of these times the star will seem to move in the vertical circle which touches its parallel.

At places between the tropics, i. e. whose latitude is less than $23^0 . 28'$, the Sun's azimuth admits of a maximum twice a day, from the time that he is vertical, or that his declination is equal to the latitude, till he again becomes vertical, after passing through the solstice.

Here $ZSP = 90^0$; $\therefore\ \cos h = \tan\Delta . \cot c = \cot\delta . \tan l.$

Ex. 7. *To find the azimuth of a terrestrial object.*

Let S (fig. 23.) be the Sun, S' the object; observe the zenith distances ZS (z), ZS' (z'), of the Sun and object, and the arc SS' (d) which their distance subtends at the observer, with an instrument such as the sextant or repeating circle, whose plane can be made to pass through them both; then

$$\sin\tfrac{1}{2}(SZS') = \sqrt{\frac{\sin\frac{1}{2}(d+z-z') . \sin\frac{1}{2}(d+z'-z)}{\sin z . \sin z'}};$$

and subtracting SZS' from the Sun's azimuth SZP, the azimuth of S' is determined.

In practice it is more usual to observe the difference of azimuth of a land object and a heavenly body with the Theodolite; then if the time be known, the azimuth of the heavenly body is known, and therefore that of the object. But if the time be unknown, then by means of a portable Transit a land object may be determined which is bisected by the middle wire at the same time as the Pole star is, when its motion is vertical; then the azimuth of that object is known; and the difference of azimuth of it and any other object may be measured by the Theodolite as before.

CHAPTER VIII.

ON REFRACTION.

219. In order to explain more simply the phenomena of the heavens, it has been taken for granted in the preceding Chapters,—that the heavenly bodies are seen in the direction of lines drawn to them from the eye of the spectator;—that on account of their great distance, it makes no difference whether they are referred to planes, passing through the place of observation, or to parallel planes drawn through the center of the Earth, i. e. that their altitudes, latitudes, &c. would be the same upon either supposition; and—that the axis round which the diurnal motion is performed, always passes through the same points of the celestial sphere. None of these assumptions are exactly true; therefore the altitudes, latitudes, &c., calculated upon the supposition of their truth, must be increased or diminished by certain quantities which are called *Corrections.*

An enumeration of the most important corrections has already been given, and the division of them into two classes, as affecting *present*, and *registered* observations, pointed out in Art. 86 of the Introduction. In this and the succeeding Chapters, each correction will be separately considered, and the formulæ belonging to it investigated; we shall begin with Refraction.

220. The atmosphere which surrounds the Earth on all sides may be considered to be made up of successive spherical strata of an elastic fluid, concentric with the Earth's surface, and *diminishing* rapidly in density as their distance from the Earth increases. Some persons have supposed the Earth's atmosphere to be of finite extent, limited by the weight of ultimate atoms of definite magnitude, no longer divisible by the repulsion of their parts; but whether its particles be infinitely divisible or not, all phenomena concur in proving that its density is quite insensible at the height of about 50 miles. By the common laws of Optics, a ray

of light moves in a straight line through any transparent medium of uniform density; and in passing from one such medium to another, unless it falls perpendicularly on their common surface, has its direction changed, and is bent, in a plane through that direction and a perpendicular to the common surface at the point of transition, towards that part of the perpendicular which falls in the denser medium.

221. To understand the consequence of this law in the case of the Atmosphere, let a plane pass through S a star, E the place of a spectator at the Earth's surface, and T the Earth's center (fig. 14.), and consequently cut all the atmospheric strata perpendicularly, their sections being represented by the spaces between the concentric circles, and each having the same uniform density throughout its whole thickness, but that density diminishing as the distance of the stratum from the Earth increases. Then a ray Sa proceeding from the star and falling upon the surface of the highest stratum of sensible density at a, is bent towards the perpendicular Ta from the direction Sax into the direction ab; again at b it is bent into the direction bc, and thus describes the polygon $abcd$ in one plane, and enters the eye in the direction dE, which consequently is the direction in which the star is seen. If we now pass to the case of nature by supposing the thickness of each stratum to be infinitely small, and the density to change by insensible degrees, the polygon becomes a curve concave to the Earth, and the apparent place s of the star lies in the direction of a tangent to the extremity of the curve described by the ray by which it is seen. Draw ES' parallel to Sa, then if there was no atmosphere ES' would be the direction of the star, but this ray describes the curve zy and never reaches the spectator at E; hence by the effect of refraction the apparent place s is *elevated in a vertical circle* above the true place S, by the $\angle sES'$, which by observation, and, as we shall shortly see, by theory also, varies nearly as the tangent of the apparent zenith ZEs, if the zenith distance be small.

222. The law just mentioned is deduced on the supposition that the Earth's surface, and the strata of the atmosphere traversed by the visual ray, are bounded by parallel

planes (which can only be true when the star is very near the zenith), and that consequently the whole refraction may be supposed to take place when the air has acquired its final density, and it errs in excess; the dimunition of refraction occasioned by the spherical shape of the strata, varies as the sum of a series involving the 3rd, 5th, &c. powers of the tangent of the apparent zenith distance. No formula is practicably applicable, if the body be observed within 10° of the horizon. The rays which proceed from a star in the zenith suffer no refraction; for inferior altitudes, the refraction increases as the altitude diminishes, and attains its maximum, about 36′, when the star is seen in the horizon.* On account of the refraction taking place wholly in a vertical circle, the azimuth of a heavenly body is not affected by it.

Effects of Refraction. Horizontal Sun. Twilight.

223. We proceed to notice some of the most remarkable phenomena caused by refraction; and first we may observe that the diurnal path of a star, being unequally raised in different parts, cannot appear accurately circular. Again, no heavenly body, independent of refraction, could be seen when it was below the horizon EH of the observer, for it would be hidden by an interposed portion of the opaque Earth; by the effect of refraction, however, when actually in the horizon, it is seen elevated above it; and it is only when the body is so far depressed, that its visual ray Sx falling on the atmosphere at x, describes a curve xE so as to have HE for its tangent, that the body appears actually in the horizon. Hence a part of the Sun is visible when his upper edge is 36′ below the horizon both at rising and setting, and the length of the day is slightly increased.

224. Also the amount of refraction near the horizon being considerable, its changes near the horizon are considerable and very rapid; so much so, that a variation of altitude of 32′, which is the Sun's apparent diameter, occasions a variation of refraction amounting to not less than 4′;

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* By the latest determination the horizontal refraction is 36′. 7″, and at zen. dist. 45° it is 57″,5; the barometer being at $27^{\text{in}}. 9,2^{\text{lines}}$ Paris meas. and the thermometer at 7°,5 Reaumur. *Schumacher's Jahrbuch,* 1837.

consequently when the Sun is in the horizon, a ray proceeding from his upper limb deviates less by 4′ than one from his lower limb, and the apparent vertical diameter is only 28′, whilst the horizontal diameter, having all its points equally raised, is very little affected, or appears to be 32′. Hence the disk of the horizontal Sun will assume a perceptibly oval shape deviating more from a circle below than above, and having its diameters to one another as 7 to 8. The same is true for the Moon in a similar situation. The distortion is of course continually varying in degree with the state of the atmosphere, and becomes insensible at considerable elevations.

When a body is near the zenith, its visual ray describes the shortest course ZE, equally through the denser and rarer strata; when near the horizon, it has a much greater space xE to traverse, and that too through the denser and less transparent strata; by reason of the absorption of light on these two accounts, bodies appear fainter as they approach the horizon.

225. We have seen that Refraction slightly lengthens the time during which we enjoy the presence of the Sun; another still more useful effect produced by it, is that pale and gradually changing light which precedes his rising and follows his setting, and prevents the abrupt transition from night to day, called *Twilight*. To understand how this is produced, let Cx represent the course of a ray within the atmosphere proceeding from a body below the horizon; when the ray arrives at x, its direction may be such that it cannot advance further into a rarer medium and must therefore be reflected; it may then describe the curve xE just as if it had come directly from a body S; and thus light, too weak indeed to produce a distinct image, may be transmitted to E from a body too far below the horizon to be visible upon it by refraction.

226. Besides this, light is in a variety of ways reflected and scattered through the atmosphere; and it is to this general dispersion that we owe our chief opportunities of sight; without it, nothing would be visible except luminous bodies, and objects so situated as to reflect rays directly from them; and the stars, instead of seeming to be fixed in an azure vault

(the colour of which is owing to the atmosphere's transmitting, or rather to the minute aqueous particles in it reflecting, the blue rays of the irregularly diffused light in greatest abundance), would appear immeasurably beyond, like brilliant spots in the midst of intense blackness. Wherever the Sun is, rays from him fall on some part of the atmosphere and enter it; and it is not impossible that however far he be below the horizon, some light is transmitted from him to the eye. The quantity of light however thus received, diminishes as the Sun's distance below the horizon is greater; and accordingly, for a certain interval before sun-rise, and after sun-set, we perceive the light of day gradually to increase in one case, and to fade away in the other; this interval called *twilight* continues during the time the Sun is invisible and not more than 18^0 below the horizon; for it is remarked that when he is less depressed, the greatest observed darkness does not prevail.

227. The duration of twilight on any day is different at different places on the Earth's surface; and at the same place at different seasons of the year it alters, and in general admits of a minimum value twice a year; for it is clear that the time of the Sun's moving vertically through 18^0 depends upon the inclination of his diurnal circle to the horizon, and upon the magnitude of that diurnal circle. When the poles are in the horizon, and the Sun in the equator, his diurnal path is a vertical *great* circle, and the duration of twilight is the time corresponding to an hour angle of 18^0, i. e. about $1^h.\ 12^m$; when he is in the tropics, it takes about $19\frac{1}{2}^0$ of his diurnal circle to be equal to 18^0 of a great circle, and the duration of twilight is about $1^h.\ 18^m$. When the poles are in the zenith, there are but two twilights in the year, lasting during the time the Sun's declination changes from 18^0 to 0^0, and from 0^0 to 18^0; i. e. about a month and a half before and after the equinoxes; so that the greatest darkness prevails only about three months.

If the greatest depression of the northern solstice below the horizon be 18^0, i.e. if the colatitude $= 23^0.\ 28' + 18^0 = 41^0.\ 28'$, or the latitude $= 48\frac{1}{2}^0$, twilight will continue all night at the summer solstice; at places in lower latitudes, for a portion of every 24^h, it will be interrupted by the greatest observed darkness; but at places in higher latitudes, twilight will continue all night from the time the Sun's midnight depression is 18^0, till it again has that value after the solstice. At London, for instance, in latitude $51\frac{1}{2}^0$, twilight continues all night from May 22nd to July 21st; it is shortest about three weeks after the autumnal, and three weeks before the vernal equinox, viz. $1^h.50^m$; at the winter solstice its duration is $2^h.7^m$; but as mist and clouds prevail most at that season, we may expect that twilight should appear shortest in midwinter.

228. We will now proceed to calculate the difference between the true and observed altitudes or zenith distances of the heavenly bodies as far as it arises from Refraction.

By the principles of Optics, if ϕ be the angle of incidence out of a denser medium into a rarer, r the refraction, and, therefore, $\phi + r$ the angle of emergence, $\sin(\phi + r) = m \cdot \sin\phi$, m being a constant depending on the difference of densities of the media, and determined by experiment. In media of uniform density, the refraction may be calculated immediately from this expression, paying attention to the form of the bounding surface; but the calculation of the refraction for the medium which surrounds the Earth, is rendered more difficult by the density of the air, not only varying at different heights, but also changing at the same place from alterations in the temperature and atmospheric pressure. The refraction at different temperatures and pressures, may be made to depend on the refraction at the mean heights of the thermometer and barometer.

229. Having given the refraction at any apparent altitude of a heavenly body for known heights of the thermometer and barometer, to find the refraction at the same apparent altitude for any other heights of the thermometer and barometer.

The refraction at a given apparent altitude of a star depends chiefly upon the density of the air at the place of observation, and is found to vary as it; if, therefore, at the same apparent altitude, r be the refraction for a standard density d, and r' that for any other density d',

$$r' = r \cdot \frac{d'}{d}.$$

The values of r are given in the tables for apparent zenith distances above 75^0 at intervals of $10'$, and for smaller zenith distances at intervals of 1^0; and the ratio of the densities can be expressed in terms of the observed heights of the thermometer and barometer as follows. Let Π be the pressure and $T^0 C$ the temperature of air when its density is d; Π', T', the pressure and temperature when its density is d'; then (Miller's Hydrostatics, Art. 36, 37.)

$$\frac{d'}{d} = \frac{\Pi'}{\Pi} \cdot \frac{1 + ET}{1 + ET'},$$

where E is the expansion of air for one degree of heat under a constant pressure, and = 0,0036438, according to Bessel (Tab. Regiom.)

Next let h be the height of the mercury in the barometer, and t its temperature corresponding to the atmospheric pressure Π, and h' and t' the height and temperature corresponding to the pressure Π'; and e the expansion of mercury for one degree of heat, = 0,00018018; then (Hydrost. Art. 32.)

$$\frac{\Pi'}{\Pi} = \frac{h'}{h} \cdot \frac{1 - et'}{1 - et};$$

$$\therefore r' = r \cdot \frac{h'}{h} \cdot \frac{1 - et'}{1 - et} \cdot \frac{1 + ET}{1 + ET'}.$$

230. *For stars near the zenith, the refraction varies as the tangent of the apparent zenith distance.*

The rays which proceed from a star near the zenith, fall nearly perpendicularly on the strata of the atmosphere, and are refracted as they would be by media, bounded by plane surfaces parallel to each other and to the horizon; therefore the refraction is the same as if the rays of light passed immediately into the last medium; that is, if *HO* (fig. 22. bis) be the horizon, *AZ* the highest stratum of atmosphere which acts upon light, *SA* the direction in which a ray from a star near the zenith is incident, *AO* the curve described, *Os* a tangent to its last direction or the direction in which the star is seen, so that *STs* is the refraction; *Os* is also the direction into which a ray SO, parallel to SA, passing directly out of a vacuum into the medium in which the eye is situated at s, would have been bent.

Let $m = \sin I \div \sin R$ out of a vacuum into the air at the Earth's surface,

$$sOZ, \text{ or the angle of emergence into the denser medium, } = \phi,$$

$$STs \text{ the refraction } = r; \quad \therefore\ SOZ = STs + sOZ = \phi + r;$$

$$\therefore\ \sin(\phi + r) = m \sin\phi,$$

$$\text{or } \sin\phi + \cos\phi \,.\, r = m\sin\phi, \text{ nearly};$$

$$\therefore\ r = (m - 1)\,.\,\tan\phi,$$

ϕ is the angle between the last direction of the ray, and a perpendicular to the horizon, or the apparent zenith distance of the star; therefore, for stars near the zenith,

$$\text{the refraction} \propto \text{tan apparent zenith distance.}$$

231. It may be proved in the same manner, whatever be the form of the common surface of two contiguous media, that the refraction of a ray which passes from one into the other $= (m - 1)$. *tan emergence into the denser medium.* The coefficient $m - 1$ is proved by experiment to vary as the difference of density of the two media; therefore, if q be the density of the air at any distance from the Earth's center, and δq the difference of density of two contiguous strata, the refraction δr in passing from the one stratum into the other,

$$= \lambda \delta q \,.\, \text{tan emergence}; \quad \therefore\ d_q r = \lambda \tan z,$$

λ being some constant. The quantity λ is found by observation to be extremely small, compared with unity, so that its square may be neglected in any calculations without sensible error.

Cassini supposed that the rays of light proceeding from a heavenly body, are refracted as they would be by an homogeneous atmosphere of the density of air at the Earth's surface, and supporting the same mercurial column as that which the atmosphere sustains. The following investigation of the refraction is upon that hypothesis.

232. *To find the refraction, considering the atmosphere homogeneous.*

Let SDZ (fig. 24.) be the course of a ray of light from a star S, which falls upon the homogeneous spherical atmosphere DO at D, and is refracted in the direction DZ to a spectator on the Earth's surface at Z. Let DZO the apparent zenith distance of $S = z$, $r =$ the refraction at D, CZ the radius of the Earth $= a$, ZO the height of the homogeneous atmosphere, $= na$, where n is very small, and nearly $= .00125$; therefore $CD = (1 + n) . a$.

Then (Art. 231.) $r = (m - 1) . \tan CDZ \ldots\ldots\ldots (1)$,

$$\text{but } \sin CDZ = \frac{CZ}{CD} . \sin z = \frac{1}{1+n} . \sin z;$$

$$\therefore \tan CDZ = \frac{\sin z}{\sqrt{(1+n)^2 - \sin^2 z}} = \frac{\sin z}{\sqrt{\cos^2 z + 2n}}$$

$$= \tan z\,(1 + 2n \sec^2 z)^{-\frac{1}{2}} \doteq \tan z\,(1 - n \sec^2 z);$$

$$\therefore r = (m - 1) \tan z\,(1 - n \sec^2 z).$$

Hence $(m - 1)(1 - 2n)$ is the refraction at apparent zen. dist. 45^0. But when barometer $= 29{,}6$ English inches, and thermometer $= 9^0{,}3$ Centigrade or $48^0{,}75$ F., it has been found that refraction at apparent zen. dist. $45^0 = 57'',595$; and therefore (Art. 229) for any other state of the atmosphere when barometer $= h'$, temp. of mercury $= t'$, temp. of air $= T'$, refraction at 45^0

$$= 57'',595 . \frac{h'}{29{,}6} . \frac{1 + 9{,}3E}{1 - 9{,}3e} . \frac{1 - et'}{1 + ET'} = (m - 1)(1 - 2n),$$

from which equation if $(m - 1)$ be determined and substituted in the above formula, we get

$$r = 59'',79 \, \frac{h'}{29{,}6} . \frac{1 - et'}{1 + ET'} . (\tan z - n \tan z \sec^2 z).$$

233. The preceding method, which resembles that in Brinkley's Astronomy, is more simple in its demonstration than any other, and gives results exact enough for all ob-

servations where the zenith distance does not exceed 80°. The numerical value of the coefficient given by Brinkley is

$$\frac{b}{29.6} \cdot \frac{1{,}0375 \,.\, 57'',82}{1 + {,}002083 \,.\, (t - 32)},$$

where t is the temperature in degrees, by Fahrenheit's thermometer, and b the height of the barometer in inches.

234. The following method of finding the refraction, from Littrow, is independent of any hypothesis on the constitution of the atmosphere, and leads to a differential equation, nearly agreeing with that of Laplace (*Mec. Cel.* Vol. IV. p. 246.), the approximate solution of which, if the coefficients are assumed properly, coincides with the preceding.

Let C (fig. 25.) be the center of the Earth, AB its surface, A the place of an observer whose zenith is Z, Aq the curve described by a ray of light proceeding from a heavenly body, AT a tangent at its extremity, in the direction of which the body will be seen. Take any point q, join Cq, and draw the tangent qZ; then, if q be the density of the spherical stratum of air at q, r the whole refraction from q to A $= \angle ATZ$, and CqT the angle of emergence at $q = z$, (by Art. 231.)

$$d_q r = \lambda \tan z\,;$$

$$\text{let } CA = 1,\ Bq = x,\ ACq = \theta,\ Cq = \rho = 1 + x,\ ZAT = Z,$$

$$\text{then } r = TZD - Z = \theta + z - Z\,;$$

$$\therefore\ d_x r = d_x\theta + d_x z = \frac{\tan z}{1 + x} + d_x z\,;\ \because \tan z = \rho\, d_\rho \theta = (1 + x)\, d_x \theta\,;$$

$$\text{but } d_x r = \lambda \tan z\, d_x q\,;$$

$$\therefore \frac{d_x z}{\tan z} = \lambda\, d_x q - \frac{1}{1 + x};$$

$$\therefore \log_e \sin z = \lambda q - \log_e (1 + x) + \text{constant.}$$

Let δ be the value of q at A, where $x = 0$;

$$\therefore \ \log_e \sin Z = \lambda\delta + \text{constant};$$

$$\therefore \ \log_e \frac{\sin z}{\sin Z} = -\lambda(\delta - q) - \log_e(1 + x) = \log_e \frac{e^{-\lambda(\delta-q)}}{1 + x};$$

$$\therefore \ \sin z = \frac{\sin Z e^{-\lambda(\delta-q)}}{1 + x};$$

$$\text{but } \ d_x r = \lambda d_x q \tan z = \frac{\lambda \sin z \, d_x q}{\sqrt{1 - \sin^2 z}};$$

consequently, substituting the value of $\sin z$,

$$d_x r = \frac{\lambda \sin Z e^{-\lambda(\delta-q)}}{\sqrt{(1 + x)^2 - \sin^2 Z e^{-2\lambda(\delta-q)}}} \, . \, d_x q.$$

This equation cannot be integrated in general, unless some hypothesis be formed as to the relation between the density of the air and its distance from the Earth, i.e. unless q be an assumed function of x; but for zenith distances less than 80^0, it may be solved in a rapidly converging series, since the greatest value of x, at which light is refracted, is known to be very small compared with the Earth's radius 1*.

In this case, neglecting the square and higher powers of λ,

$$\begin{aligned} d_x r &= \frac{\lambda \, . \, \sin Z d_x q}{\sqrt{1 + 2x + x^2 - \sin^2 Z}} \\ &= \frac{\lambda \, . \, \sin Z d_x q}{\sqrt{\cos^2 Z + 2x + x^2}} \\ &= \frac{\lambda \tan Z d_x q}{\sqrt{1 + (2x + x^2)\sec^2 Z}} \quad \ldots\ldots\ldots (2) \\ &= \lambda \, . \tan Z d_x q \, . \, (1 - x \sec^2 Z), \end{aligned}$$

neglecting x^2, &c.;

$$\therefore \ r = \int_x \lambda \tan Z d_x q \, . \, (1 - x \sec^2 Z),$$

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* According to Vince, (*Astronomy*, Chap. VII.) the atmosphere does not act upon light at a greater altitude than 77,25 miles; which, divided by 4000, gives the greatest value of $x = .01931$.

the integral being taken from the greatest value of x to $x = 0$, and consequently from $q = 0$ to $q = \delta$.

The integral of $\lambda \tan Z d_x q$ between these limits

$$= \lambda \tan Z . \delta ;$$

$$\text{and } \textstyle\int_x x d_x q = qx - \int_x q,$$

of which, the part qx vanishes between the limits, and the part $\int_x q =$ the atmospheric pressure, or the weight of the mercury in the barometer, neglecting the variation of gravity within the limits of x; therefore, if b be the height of the mercury, considering its density unity,

$$r = \lambda \delta \tan Z . \left\{ 1 - \frac{b}{\delta} . \sec^2 Z \right\}.$$

The mean value of δ, if the density of mercury is 1, $= \frac{1}{10485}$, and the mean value of $b : 1 :: 29.60$ inches : 4000 miles nearly; therefore, the mean value of $\frac{b}{\delta} = .0012517$, which may be used without sensible error for the coefficient of $\sec^2 Z$; therefore, substituting this value,

$$r = \lambda \delta . \left\{ \tan Z - ,0012517 . \tan Z . \sec^2 Z \right\},$$

or if, at A, height of barometer $= h'$, temp. of mercury $= t'$, temp. of air $= T'$,

$$r = \lambda h' . \frac{1 - et'}{1 + ET'} . \left\{ \tan Z - ,0012517 . \tan Z . \sec^2 Z \right\},$$

where the coefficient λ must be determined by observation.

If the expansion of equation (2), be carried on as far as x^2, the refraction

$$= \lambda h' . \frac{1 - et'}{1 + ET'} . \tan Z \left\{ 1 - . \frac{,0012517}{\cos^2 Z} + ,00000139 . \frac{2 + \sin^2 Z}{\cos^4 Z} \right\}.$$

235. *To determine the coefficient of refraction from observations of circumpolar stars.*

Let z, z' be the greatest and least observed zenith distance of the same circumpolar star, r, r' the refractions, and c the colatitude of the place of observation; then, if a be the required coefficient of refraction, which is the quantity designated in the last article by λ, and ascertained nearly to equal $59'',8$, $r = au$, u representing in this equation a known function depending on the observed zenith distance, and the observed height of the thermometer and barometer; similarly, $r' = au'$, u' being the corresponding function for the zenith distance z'; therefore, because $z + r$ and $z' + r'$ are the true zenith distances of the star, (as in Art. 153.),

$$\therefore\ 2c = z + r + z' + r'$$

$$= z + z' + a . (u + u').$$

If z_1, z_1', u_1, u_1', are corresponding values of z, z', u, u' for another circumpolar star,

$$2c = z_1 + z_1' + a . (u_1 + u_1');$$

$$\therefore\ z + z' + a . (u + u') = z_1 + z_1' + a . (u_1 + u_1');$$

$$\therefore\ a = \frac{z_1 + z_1' - z - z'}{u + u' - u_1 - u_1'}.$$

The numerical value of a determined by numerous observations, is, as was before observed, $= 59'',8$.

236. Bradley made use of the solstitial meridian zenith distances of the Sun in determining the coefficient of refraction. If s and s' be these distances, r, r' the refractions, ω the obliquity, and l the latitude of the place of observation, (as in Art. 172.),

$$l - \omega = s + r,$$

$$l + \omega = s' + r';$$

$$\therefore\ 2l = s + s' + r + r',$$

$$\text{or } 180^0 - 2c = s + s' + a . (\sigma + \sigma'),$$

σ and σ' being the known functions of s and s', but, by the preceding Art. $2c = z + z' + a . (u + u')$;

$$\therefore\ 180^0 = s + s' + z + z' + a . (\sigma + \sigma' + u + u'),$$

$$\text{and } a = \frac{180^0 - s - s' - z - z'}{\sigma + \sigma' + u + u'}.$$

Bradley calculated his refractions from the equation $r = 57''. \tan(z - 3r')$, determining r' by the equation $r' = 57''. \tan z$. He never gave any demonstration of his formula, and most probably obtained it by conjecture and trial. From Cassini's hypothesis, (Art. 232) an equation of the same form may be obtained. See *Delambre*, Chap. XIII.

237. If the declination of any one star were known correctly, a table of refraction might be easily formed, by observing its altitude and azimuth at the same instant. From the time elapsed between the observation and the star's transit over the meridian, the hour angle of the star may be found; and from this hour angle, the star's declination, and the observed azimuth, the true zenith distance of the star, at the instant of observation, may be obtained: having thus found the true and the observed zenith distance, the corresponding refraction is known. In this manner the accuracy of any table of refraction may be ascertained.

The refraction found by the preceding methods cannot be relied on, when the zenith distance is greater than 80⁰; beyond this limit, it does not depend solely on the state of the atmosphere at the place of observation, but sometimes when the zenith distance is near 90⁰, varies at the same place on different days by 3′ or 4′, whilst the temperature and barometric pressure remain unaltered.

238. The propositions of the preceding Chapters have been demonstrated without taking into consideration the refractive power of the atmosphere; therefore the altitudes and zenith distances which enter into them, are not the results of direct observation, but are formed by correcting the observed altitudes and zenith distance for the error of refraction, by tables calculated from formulæ similar to those in this Chapter.

Since refraction causes the heavenly bodies to appear elevated above their true places, it accelerates their rising, and retards their setting; but, as the azimuth is not altered by it, the times of transit over the meridian are the same as if there were no refraction.

239. *To find how much the time of a star's rising is altered by refraction.*

Let $h = ZPS$ be the hour angle when the star really rises,

$h' = ZPs$ .... appears to rise, (fig. 10. bis),

or when the zenith distance $= 90^0 +$ the horizontal refraction (r).

$$\text{Then } h' - h = \frac{Ss}{\cos\delta} = \frac{r}{\cos\delta\cos PSH} = \frac{r}{\sqrt{(\cos\delta)^2 - (\sin l)^2}},$$

$$\because\ Ss\cos PSH = sO = r,$$

and $\sin l = \cos\delta\sin PSH$ from the right-angled triangle PSH;

$$\therefore\ h' - h = \frac{r}{\sqrt{\cos(l+\delta)\cos(l-\delta)}},$$

from which the acceleration of rising $\dfrac{h' - h}{15^0}$ is known.

On account of the smallness of r, this equation gives the acceleration in the time of rising with sufficient accuracy; h and h' may however be determined exactly from the formulæ

$$\cos h = -\tan l\,.\,\tan\delta,$$

$$\cos h' = -\sin r\,.\,\sec l\,.\,\sec\delta - \tan l\,.\,\tan\delta,$$

derived from the triangles ZPS, ZPS', where $ZS' = 90^0 + r$; and then $h' - h$ may be found.

240. Let T and T' be the times during which a star nearly in the equator, is above and below the horizon; then $\frac{1}{2}(T - T')$ is the error in the time between rising and culminating caused by refraction, for without refraction, T would equal T'; also Ss or $\angle SPs = \dfrac{r}{\cos PSH}$, since $\delta = 0$;

$$\therefore \frac{T - T'}{4} = \frac{1}{15^0} \cdot \frac{r}{\cos l};$$

$$\therefore \quad r = \tfrac{1}{4} \cdot 15^0 \cdot (T - T') \cdot \cos l.$$

The mean value of r, found in this way, is $33'$.

241. Exactly in the same way, the duration of Twilight may be found.

If h' and h be the hour angles corresponding to the beginning and end of twilight, and l and δ as before,

$$\cos h = - \tan l \cdot \tan \delta,$$

$$\cos h' = - \sin 18^0 \cdot \sec l \cdot \sec \delta - \tan l \cdot \tan \delta;$$

from which h and h', and the difference $h' - h$, may be deduced.

In summer the greatest depression of the Sun below the horizon = the distance of the equator from the horizon − the Sun's declination $= 90^0 - l - \delta$; therefore twilight will continue all night, if

$$90^0 - l - \delta \text{ is less than } 18^0, \text{ or } l + \delta > 72^0.$$

If $l + \delta = 72^0$, the evening twilight will end, when that of the morning begins.

242. *To find the alteration produced by refraction on any diameter of the Moon.*

The differences of refraction for the small extent of the lunar disk may be considered proportional to the differences of altitude; therefore, if the horizontal diameter of the Moon be taken for the axis of x, the ordinates perpendicular to it will be diminished in proportion to their lengths, and the circular disk will become an ellipse.

Let Δ be the Moon's apparent horizontal diameter; and therefore $\Delta - n$ the vertical diameter, n being the difference of refraction for a difference of altitude Δ, then the equation to the ellipse is

$$y^2 = \frac{(\Delta - n)^2}{\Delta^2} \cdot \{\tfrac{1}{4} \cdot \Delta^2 - x^2\}.$$

If $2R$ be any diameter inclined at an angle a to the horizontal diameter, $y = R \sin a$, and $x = R \cos a$;

$$\therefore\ R^2 \sin^2 a = \tfrac{1}{4} . (\Delta - n)^2 - \frac{(\Delta - n)^2}{\Delta^2} . R^2 \cos^2 a;$$

$$\therefore\ 2R = \frac{(\Delta - n) . \Delta}{\sqrt{\Delta^2 \sin^2 a + (\Delta - n)^2 \cos^2 a}}$$

$$= \frac{\Delta^2 - \Delta n}{\sqrt{\Delta^2 - 2\Delta n \cos^2 a}}, \text{ nearly,}$$

$$= \Delta . \left(1 - \frac{n}{\Delta}\right) . \left\{1 + \frac{n}{\Delta} \cos^2 a\right\}$$

$$= \Delta . \left\{1 - \frac{n}{\Delta} \sin^2 a\right\}.$$

But if δ (fig. 26.) be the true horizontal diameter LM of the Moon, the extremities of which L and M are raised by refraction to l and m in the verticals ZL, ZM, then $lm = \Delta$. Draw ZO bisecting LM and lm, and let $Zo = z$, then $Oo = r$ = refraction at apparent zenith distance z;

$$\therefore\ \Delta = \frac{\delta \sin z}{\sin (z + r)} = \frac{\delta}{1 + r \cot z} \text{ nearly;}$$

$$\therefore\ 2R = \frac{\delta}{1 + r \cot z} \left\{1 - \frac{n}{\delta} . (1 + r \cot z) . \sin^2 a\right\}$$

$$= \delta . (1 - r \cot z) . \left\{1 - \frac{n}{\delta} \sin^2 a\right\}, \text{ nearly.}$$

The use of this formula is to correct an observed diameter of the Moon, or an observed distance of the cusps in a solar eclipse; $2R$ may also represent the apparent small distance between two stars observed with a Micrometer, and then δ will be their true distance.

CHAPTER IX.

ON PARALLAX.

Nature and variation of Parallax, and mode of finding it.

243. Hitherto no account has been taken of the place of the observer; only it has been considered to coincide with the Earth's center, because the distance of by far the greater part of the heavenly bodies is so great, that the Earth's diameter compared with that distance is evanescent. There are however several bodies, as the Sun, Moon, and Planets, so near to us, that the Earth's dimensions in regard to them cannot be neglected;—bodies, whose apparent places relative to the fixed stars are different to observers at different stations, and also different from their true or *geocentric* places, i. e. such as would be seen from the Earth's center; for since the Earth's center is nearly at the same distance, and in a known direction from every point of its surface, and since it is through that point that we have supposed the axis of the heavens and all the great circles of the sphere to pass, it has been agreed to consider that as the *true* place of a heavenly body which it would be seen to occupy among the fixed stars from the Earth's center. Hence it becomes necessary to assign, for all distances and positions of these bodies, this difference, called *parallax*, between their true and apparent places; in order that observations made at different places may be generalized, and put into a state of being compared with one another, by being reduced to what they would have been if made from the center. As was before stated, the correction of parallax is applied, not because we see the body in a point of space which it really does not occupy, as in the instance of refraction, but because we do not see it in the position it ought to occupy, conformable to previous calculations and suppositions, and so as to be convenient for comparison with other observations; the manner in which its amount is to be ascertained we proceed to explain.

244. Let O fig. 15, plate 1, be an observer, S a heavenly body, and T the Earth's center; then the body's apparent zenith distance is $\angle ZOS$, and its true zenith distance $\angle ZTS$, less than the former by the angle OST which is subtended by the Earth's radius at S, and is called the parallax of S. From O, the body is seen among the fixed stars at a, and from T it is seen at b; hence parallax, like refraction, takes place in a vertical circle, but it makes the apparent place *lower* than the true place. Also $\sin OST = \sin ZOS \times \frac{OT}{TS}$; therefore, for the same body, since we may suppose its distance from the Earth's center not to sensibly alter whilst passing from the meridian to the horizon, the sine of the parallax in altitude varies as the sine of the apparent zenith distance. Since it has a course of variation comprised within the time the body is above the horizon, this correction is called the *diurnal* parallax.

245. *To find the parallax in altitude of a known body.*

Let TO, the distance of the observer from the center of the terrestrial spheroid $= r$, $TS = R$, ZTS the true zenith distance $= z$, $OST = p$, and ZOS the apparent zenith distance $= z + p = z'$;

$$\therefore \sin p = \frac{r}{R} \cdot \sin(z + p) \dots\dots\dots (1)$$

$$= \frac{r}{R} \cdot (\sin z \cdot \cos p + \cos z \cdot \sin p);$$

therefore, dividing by $\cos p$,

$$\tan p = \frac{r}{R} \cdot \frac{\sin z}{1 - \frac{r}{R} \cdot \cos z} \dots\dots\dots (2).$$

246. The parallax is generally so small that, except for the Moon, no sensible error is introduced by making $\sin p = p$; therefore, from (1),

$$p = \frac{r}{R} \cdot \sin z'.$$

Hence p diminishes as R increases, and although p can never be really 0, yet if R exceed $\frac{r}{\sin 1''}$, p must be less than $1''$, and would be too small to admit of detection by our instruments. The fixed stars have no sensible parallax; and in this the effect of parallax differs from that of refraction, that the latter is independent of the distance of the body observed, whilst the former diminishes as the distance increases, and at length becomes insensible.

247. When the body is seen in the horizon, $z + p = 90^0$; if therefore P be the value of p in this case, $\sin P$ or $P = \frac{r}{R}$;

$$\therefore \sin p = \sin P . \sin z',$$

$$\text{or } p = P . \sin z' \ldots\ldots\ (4),$$

which is the parallax at the apparent zenith distance z', in terms of the horizontal parallax and that zenith distance.

We can also expand the parallax in altitude in terms of the true zenith distance and horizontal parallax; for since $\sin p = \sin P . \sin (z + p)$, by a common process we get

$$p = \sin P \sin z + \tfrac{1}{2} \sin^2 P \sin 2z + \tfrac{1}{3} \sin^3 P \sin 3z + \&c.$$

248. If P' be the horizontal parallax at the equator, and a the radius of the equator,

$$P' = \frac{a}{R}; \ \therefore P = \frac{r}{a} . P' = P' . \sqrt{1 - e^2 \sin^2 l},$$

(Art. 126), P being the horizontal parallax at a place in latitude l.

Parallax, as we perceive, is greatest when the body is in the horizon; and the sine of the *horizontal* parallax equals the Earth's radius divided by the body's distance from the Earth's center. Hence the horizontal parallax will vary with the latitude of the observer if the Earth be not spherical, for in that case, the Earth's radius will diminish from the equator to the pole; but at the Moon's mean distance, it is found that her horizontal parallax varies in different latitudes; this proves the spheroidal figure of the Earth.

Again, during the time between culminating and setting, the Moon's distance from the Earth's center continues nearly the same; hence in the triangle OTS, OT and TS remaining constant, OS will increase with the zenith distance; and therefore the apparent diameter will diminish from the time the Moon is on the meridian, till she sets, at which time it will differ least from the true apparent diameter observed from the center; of course the effect is the same for the Sun, but too small by reason of his vast distance to be of practical importance; but though it is thus unquestionable that both the Sun and Moon are seen under the least angle when in the horizon, yet by reason of their being then referred to a part of the heavens at a greater apparent distance, their disks appear larger in the horizon than in the zenith.

249. *To find the effects of parallax on the hour angle and declination of a known body, i.e. to find the errors with which they are affected, if computed from an observed zenith distance.*

Let Z (fig. 29.) be the zenith of any place, P the pole of the equator ♈MI, S the true place of a heavenly body, depressed by parallax to S' in the vertical ZS. Draw SO perpendicular to PS', and let a (SPS') be the parallax in the hour angle ZPS (h), and β ($S'O$) the parallax in the declination SI (δ); make $S'Q = 90°$, producing $S'P$ if necessary, and join ZQ by the arc of a great circle.

$$\text{Then } a = \frac{SO}{\sin PS} = \frac{SS' . \sin ZS'P}{\sin PS}$$

$$= p . \frac{\sin ZS' . \sin ZS'P}{\sin PS}$$

$$= p . \frac{\sin ZP . \sin ZPS'}{\sin PS}$$

$$= p . \frac{\cos l . \sin (h + a)}{\cos \delta};$$

$$\beta = S'O = SS' \cos SS'O = P \sin ZS' \cos ZS'Q$$

$$= P \cos ZQ, \quad \text{from the triangle } ZS'Q \text{ in which } S'Q = 90^{\circ},$$

$$= P (\cos ZP \,.\, \cos PQ + \sin ZP \sin PQ \cos ZPQ)$$

$$= P \left\{\sin l \cos \delta - \cos l \sin \delta \cos (h + a)\right\}.$$

Let α' be determined by the equation

$$\alpha' = P \,.\, \frac{\cos l \,.\, \sin h}{\cos \delta};$$

then hour angle affected by parallax – true hour angle

$$= P \,.\, \frac{\cos l \,.\, \sin (h + \alpha')}{\cos \delta},$$

declination affected by parallax – true declination

$$= - P \,.\, \left\{\sin l \cos \delta - \cos l \,.\, \sin \delta \,.\, \cos (h + \alpha')\right\}.$$

250. The error in the right ascension of S caused by parallax, is the same as that of the hour angle; and $ZPS = ♈ I - ♈ M =$ the right ascension of S (a) – the right ascension of the mid-heaven (A); therefore

R. A. affected by parallax – true R. A.

$$= P \,.\, \frac{\cos l \,.\, \sin (a - A)}{\cos \delta}.$$

251. To find the effects of parallax on the latitude and longitude of a heavenly body.

If P be the pole of the ecliptic, $S'O$ is the parallax in latitude, and SPS' the parallax in longitude of the star S; therefore, the parallax in longitude

$$= P \,.\, \frac{\sin ZP \,.\, \sin ZPS}{\sin PS};$$

and the parallax in latitude

$$= P \,.\, \left\{\cos ZP \,.\, \sin PS - \cos PS \,.\, \sin ZP \,.\, \cos ZPS\right\}.$$

In this case, $ZP =$ the height of the nonagesimal (n), and ZPS or $ZPS' =$ the longitude of $S\,(l)$ – the longitude of the nonagesimal (L); therefore, if λ be the body's latitude, longitude affected by parallax – true longitude

$$= P\,.\,\frac{\sin n\,.\,\sin (l - L)}{\cos \lambda};$$

latitude affected by parallax – true latitude

$$= -\,P\,\{\cos n \cos \lambda - \sin n\,.\,\sin \lambda\,.\,\cos (l - L)\}.$$

252. *To find the augmentation by parallax of the Moon's diameter.*

Let D be the apparent diameter of the Moon p (fig. 28.), seen from the center of the Earth, and D' the apparent diameter seen from O, then since the body is the same,

$$D' = D\,.\,\frac{Cp}{Op} = D\,.\,\frac{\sin z'}{\sin (z' - p)};$$

$$\therefore\ D' - D = D\,.\,\frac{\sin z' - \sin (z' - p)}{\sin (z' - p)};$$

$$= 2D\,.\,\frac{\sin \tfrac{1}{2}p\,.\,\cos (z' - \tfrac{1}{2}p)}{\sin (z' - p)}.$$

If Δ be the value of D' when the Moon is in the horizon, in which case $z' = 90^0$ and $p =$ the horizontal parallax P,

$$\Delta = \frac{D}{\cos P}; \quad \therefore\ D = \Delta \cos P.$$

The same formulæ may be applied to the Sun or any of the planets, but these latter bodies are too distant for parallax to produce any sensible effect on their apparent diameters.

253. *To find the true latitude and longitude of a heavenly body, from the latitude and longitude affected by parallax, by a more exact method.*

Let l, λ, r, be the longitude, latitude, and distance of a heavenly body as seen from the Earth's center, l', λ', r' the corresponding quantities from the surface; Λ and L the latitude and longitude of the observer's zenith, and R his distance from the Earth's center, which $= r' \sin P$. Let the planet's place be referred to rectangular co-ordinates x, y, z, the center of the Earth being the origin, and the axis of x passing through the vernal equinox, and that of z through the pole of the ecliptic. Similarly, let X, Y, Z, be the co-ordinates of the observer's place, and let x', y', z' be the co-ordinates of the planet, referred to parallel axes passing through the place of observation. Then

$$x = r \cos\lambda . \cos l, \quad y = r \cos\lambda . \sin l, \quad z = r \sin\lambda \ldots (1),$$

$$x' = r' \cos\lambda' \cos l', \quad y' = r' \cos\lambda' \sin l', \quad z' = r' \sin\lambda' \ldots (2),$$

$$X = R \cos\Lambda . \cos L, \quad Y = R \cos\Lambda . \sin L, \quad Z = R \sin\Lambda \ldots (3),$$

$$= r' \sin P \cos\Lambda . \cos L, \quad = r' \sin P . \cos\Lambda . \sin L, \quad = r' \sin P . \sin\Lambda ;$$

$$\therefore \; x = x' + X = r' . \{\cos\lambda' \cos l' + \sin P . \cos\Lambda . \cos L\},$$

$$y = y' + Y = r' . \{\cos\lambda' \sin l' + \sin P . \cos\Lambda . \sin L\},$$

$$z = z' + Z = r' . \{\sin\lambda' + \sin P . \sin\Lambda\}.$$

But, from equations (1),

$$\tan l = \frac{y}{x}, \quad \tan\lambda = \frac{z}{x} . \cos l;$$

$$\therefore \; \tan l = \frac{\cos\lambda' \sin l' + \sin P . \cos\Lambda . \sin L}{\cos\lambda' \cos l' + \sin P . \cos\Lambda . \cos L},$$

$$\tan\lambda = \frac{\{\sin\lambda' + \sin P . \sin\Lambda\} . \cos l}{\cos\lambda' \cos l' + \sin P . \cos\Lambda . \cos L}.$$

From the first of these formulæ l may be found, and then λ from the second; and these being known, r may be obtained by the equation

$$r = \frac{x}{\cos\lambda . \cos l} = r' . \frac{\{\cos\lambda' . \cos l' + \sin P . \cos\Lambda . \cos L\}}{\cos\lambda . \cos l}.$$

254. If D be the apparent diameter of the body seen from the center of the Earth, D' the diameter at the place of observation,

$$D = \frac{r'}{r} \cdot D' = \frac{D' \cdot \cos\lambda \cdot \cos l}{\cos\lambda' \cdot \cos l' + \sin P \cdot \cos\Lambda \cdot \cos L}.$$

Λ, the latitude of the observer's zenith, is the same as the zenith distance of the nonagesimal, and L, the longitude of the zenith, the same as the longitude of the nonagesimal.

255. In the same manner, if a and δ be the right ascension and declination of the planet as seen from the center of the Earth, a' and δ' at the place of observation, A the right ascension of the zenith, or of the mid-heaven, and D the declination of the place of observation as seen from the center of the Earth, which is equal to the latitude of the place of observation;

$$\tan a = \frac{\cos\delta' \cdot \sin a' + \sin P \cdot \sin A \cdot \cos D}{\cos\delta' \cos a' + \sin P \cdot \cos A \cdot \cos D},$$

$$\tan\delta = \frac{\{\sin\delta' + \sin P \cdot \sin D\} \cos a}{\cos\delta' \cdot \cos a' + \sin P \cdot \cos D \cdot \cos A}.$$

256. Many methods have been proposed for determining the amount of the parallax of those heavenly bodies for which it is of sensible magnitude; some of them being of great labour and requiring observations at distant places. Thus let O, o, (fig. 15.) be two observers on the same meridian, on which also is the body S; and let its zenith distances ZOS, zoS be observed; then the $\angle\, OTo$ which is the sum or difference of the latitudes of the places, and the radii TO, To being also known, both TS and the angles OST, oST, which are the parallaxes at the two stations, may be determined. In this manner the parallax of the Moon, and of the planet Mars, was ascertained by observations at the Cape of Good Hope, and at Stockholm, places nearly on the same meridian on different sides of the equator. For all the heavenly bodies however, except the Moon, the amount of parallax is exceedingly small, and can only be satisfactorily ascertained on the occurrence of rare phenomena such as the the transit of a planet over the Sun's disk; but the Moon's parallax can be determined with tolerable exactness by observations made at the same place, or at two distant places in the way just stated; and we shall now investigate the formulæ for that purpose.

257. *To measure the parallax of a heavenly body, by observations made in the plane of the meridian.*

Let $ZAP(z)$ (fig. 30.), $Z'A'P(z')$ be two meridian zenith distances of the Moon's center P, determined simultaneously at two known places A and A', on the same meridian, by measuring the difference of altitude of the Moon's limb and a known star nearly in contact with it; as the zenith distance of the star is unaffected by parallax, when corrected for refraction it will at once lead to the zenith distance of the Moon affected only by parallax. Take C the center of the Earth, draw the lines CP, CAa, $CA'a'$, and let $CA = r$, $CA' = r'$, $CP = R$, $l =$ the latitude of A, $l' =$ the latitude of A'; then, if ϕ, ϕ' be the known angles between the normals at A and A', and the lines CA, CA' (Art. 129), $PAa = z - \phi$, $PA'a' = z' - \phi'$; therefore, if p, p', be the parallaxes at A and A', we have

$$\text{from the triangle } CAP \ldots\, p = \frac{r}{R} . \sin(z - \phi),$$

$$\ldots\ldots\ldots\ldots\ldots\, CA'P \ldots\, p' = \frac{r'}{R} . \sin(z' - \phi');$$

$$\therefore p + p' = \frac{r . \sin(z - \phi) + r' . \sin(z' - \phi')}{R}.$$

But $p + p' = APA'$

$$= PAa + PA'a' - ACA'$$

$$= z - \phi + z' - \phi' - (l - \phi + l' - \phi')$$

$$= z + z' - l - l';$$

$$\therefore R = \frac{r . \sin(z - \phi) + r' . \sin(z' - \phi')}{z + z' - l - l'};$$

therefore, if P' be the equatoreal parallax, and a the radius of the Earth's equator,

$$P' = a . \frac{z + z' - l - l'}{r \sin(z - \phi) + r' \sin(z' - \phi')}, \text{ (Art. 248.)}$$

A and A' are supposed on different sides of the equator; if they are on the same side, l' is negative.

258. If p and p' are too large to be considered equal to their sines, we have

$$\sin p = \frac{r}{R} \,.\, \sin (z - \phi) \,.\,.\,.\,.\,.\,.\,.\, (1),$$

$$\sin p' = \frac{r'}{R} \,.\, \sin (z' - \phi'),$$

and calling $z + z' - l - l' = m$, or making $p' = m - p$,

$$\sin (m - p) = \frac{r'}{R} \,.\, \sin (z' - \phi'),$$

therefore, $\sin m \cos p - \cos m \sin p = \dfrac{r'}{R} \sin (z' - \phi')$,

and dividing this by equation (1);

$$\sin m \,.\, \cot p - \cos m = \frac{r'}{r} \,.\, \frac{\sin (z' - \phi')}{\sin (z - \phi)};$$

therefore, $\tan p = \dfrac{r \,.\, \sin (z - \phi) \,.\, \sin m}{r' \,.\, \sin (z' - \phi') + r \,.\, \cos m \,.\, \sin (z - \phi)}$.

Whence p may be found, and thence

$$R = r \,.\, \frac{\sin (z - \phi)}{\sin p}, \text{ and therefore } P' = \frac{a}{R}.$$

259. By this method, the parallax of a heavenly body, the declination of which is unchangeable, may be found if the observers are on different meridians; but if the declination and, consequently, the meridian zenith distance changes, as is the case with all the bodies which have a sensible parallax, the meridian zenith distance at A' must be observed for several successive days, from which, by the method of interpolations, we may find the meridian zenith distance as seen from a place in the same latitude as A' at the time when the body was on A's meridian; that is, the meridian distance which it would have at A', if it were to move from the meridian of A to that of A', without changing its declination.

260. *To find the Moon's parallax, by observations made out of the plane of the meridian.*

Let l be the latitude of the observer, δ the declination of the Moon, h the hour angle calculated from an observed zenith distance, and which is therefore affected with the error of parallax

$$a = P \, . \, \frac{\cos l \, . \sin h}{\cos \delta}. \text{ (Art. 249.)}$$

Let a', h', be the values of a and h, corresponding to another observation; then, as before,

$$a' = P \, . \, \frac{\cos l \, . \sin h'}{\cos \delta},$$

neglecting the small change of declination between the observations;

$$\therefore a - a' = P \, . \cos l \, . \, \frac{\sin h - \sin h'}{\cos \delta}$$

$$= 2P \, . \, \frac{\cos l}{\cos \delta} \, . \sin \tfrac{1}{2} (h - h') \, . \cos \tfrac{1}{2} (h + h');$$

now h and h' are known from the observed zenith distances, and therefore $h - h'$ the apparent decrement of the hour angle in the time (t) between the observations; let m be the increase of the Moon's right ascension in the time t, furnished by the tables of her motion, then the real decrement of the hour angle between the observations $= 15^0 \, . \, t - m, =$, also $h - a - (h' - a')$;

$$\therefore a - a' = h - h' - (15^0 \, . \, t - m) \text{ is known;}$$

therefore we know

$$P = \tfrac{1}{2} \, . \, \frac{(a - a') \, . \cos \delta}{\sin \frac{1}{2} (h - h') \, . \cos \frac{1}{2} (h' + h) \, . \cos l}.$$

If the observations are made on different sides of the meridian,

$$P = \tfrac{1}{2} \, . \, \frac{(a + a') \, . \cos \delta}{\sin \frac{1}{2} (h + h') \, . \cos \frac{1}{2} (h' - h) \, . \cos l}.$$

261. The quantities h and h' must be calculated from the observed zenith distances corrected for refraction; this correction may be avoided by ascertaining, at each of the observations, the zenith distance of a star nearly in contact with the Moon, which will be affected with the same error from refraction as the corresponding zenith distance of the Moon is, but will be unaffected by parallax. Let then the computed hour angle of the star at the first observation be $h_{,}$, and at the second $h_{,}'$, and let r, r', be the errors of refraction; ∴ the true hour angles are $h_{,} + r$, $h_{,}' + r'$, and their difference

$$= h_{,} - h_{,}' + (r - r') = d + 15^0 . t,$$

d being the difference of right ascension of the stars. (Art. 158.) But the true hour angles of the Moon at the first and second observations are $h - a + r$, $h' - a' + r'$, and their difference

$$= h - h' - (a - a') + r - r' = -m + 15^0 . t;$$

hence, subtracting this equation from the former,

$$(h_{,} - h_{,}') - (h - h') + (a - a') = d + m,$$

which gives $(a - a')$.

The two preceding methods enable us to determine the Moon's parallax with considerable accuracy; they may also be applied to the planets; but the Sun's parallax, which is one of the most important elements of Astronomy, is too small to be discovered in this way, as a slight error of observation may entirely vitiate the result. An accurate mode of determining the solar parallax, is described in a subsequent Chapter.

Distances of the heavenly bodies deduced from parallax.

262. The most interesting results deduced from the observations and theory of parallax are the *distances* and *magnitudes* of the heavenly bodies. We have seen that the sine of the horizontal parallax is equal to the Earth's radius divided by the distance of the body; hence the distance of the body equals the Earth's radius divided by its horizontal parallax $= \dfrac{57^0.29578}{P^0} \times$ Earth's radius, where $\dfrac{P}{57.29578}$ represents the

circular measure of the horizontal parallax and is substituted for its sine, since it is very small, 57°,29578 being the number of degrees in an arc of a circle whose length is equal to the radius. Now the Sun's distance is variable and the Moon's also, as we have seen, therefore the eq. horizontal parallax both of the Sun and Moon varies; taking however the mean values, they are respectively 8″.57116 or 0°,00238, and 57′.1″,8 or 0°,9505; hence the Sun's distance $= \frac{57,29578}{,00238} \times$ Earth's eq. radius = 24074, or nearly 24 thousand times the Earth's radius, i.e. about 96 millions of miles; and the Moon's distance $= \frac{57,29578}{,9505}$ × Earth's eq. radius = 60,2796 times the Earth's radius.

As the Sun's horizontal parallax is so small, a very slight error in it, causes a great error in determining his distance; but an error of 1″ in the Moon's parallax does not affect her distance with an error exceeding 70 miles.

263. Again since the apparent radius of a body is equal to its real radius divided by its distance, we have the real radius $= \frac{\text{apparent radius}}{\text{horizontal parallax}} \times$ Earth's radius; this gives the Moon's diam ~~radius~~ about ,2723 or $\frac{3}{11}$ths of the Earth's, and the Sun's about 112,15 or 112 times the Earth's; taking the apparent $\frac{1}{2}$ diameter of the Sun and Moon, at their mean distances from the Earth, at 16′.0″,9 and 15′.31″,95 respectively. Hence if the Sun's center coincided with the Earth's, his body would include the whole of the lunar orbit, and extend almost as far again beyond it. Also the Moon's volume is about $\frac{1}{49}$th part of the Earth's volume, and the Sun's between 13 and 14 hundred thousand times that quantity.

As the horizontal refraction never exceeds 36′, the Moon is more depressed by parallax than raised by refraction.

Annual Parallax.

264. Though the Earth's radius bear no assignable ratio to the distance of the fixed stars, yet that may not be the case with respect to the radius of her orbit which is 24000

times as great; consequently the apparent places of some of the stars, though unaffected by the diurnal parallax, may not be the same for all positions of the Earth in her orbit. The angular distances of a star from the line joining the Earth and Sun, as observed from the Earth and Sun, will differ from one another by the angle subtended by the radius of the Earth's orbit at the star, which, since it runs through all its variations in the course of a year, is called the star's *annual* parallax; whether this angle has a perceptible magnitude even for the nearest of the fixed stars seems doubtful; by determining the value which it cannot exceed, we ascertain the limit of the proximity of the nearest fixed stars to be at least eighty thousand times the Sun's distance from the Earth.

Astronomers have sought with great care to find by observations, if any of the stars have a sensible annual parallax; and we shall now give the method employed for that purpose.

265. *To find the annual parallax of a fixed star in latitude and longitude.*

Let S be the Sun in the center of the sphere, (fig. 29. bis) E the Earth in its orbit, Z its place referred to the ecliptic, K the pole of the ecliptic, s a fixed star whose true angular distance from the line ES is $\angle sSZ$, but apparent angular distance $\angle sEZ$. Therefore in the plane ESs drawing Ss' parallel to Es, s' is the apparent place of the star; and drawing so perpendicular to the circle of latitude through s', os' is the annual parallax in latitude, and $\angle sKs'$ that in longitude. Now $\dfrac{\sin SsE}{\sin ESs} = \dfrac{ES}{Es}$;

$\therefore$ $\sin SsE$ or $\angle sSs' = \dfrac{ES}{Es} \sin Zs \propto \sin Zs$ nearly; for ES is nearly constant, and the star being at so immense a distance from the Earth, any alteration produced in that distance by the Earth's change of position must be insensible. Let ϕ equal greatest value of sSs', i. e. when Zs becomes $= 90^0$; $\therefore \phi = \dfrac{ES}{Es}$ and $sSs' = \phi \sin Zs$. Let the longitude and latitude of the star be l, λ, and the longitude of the Earth seen from the Sun be $\oplus$; $\therefore LZ = l - \oplus$. Hence longitude affected by parallax – true long. $= \dfrac{so}{\sin Ks} = \dfrac{ss' \sin Kss'}{\sin Ks}$

$$= \frac{\phi \sin Zs \sin LsZ}{\cos Ls} = \frac{\phi \sin ZL}{\cos Ls} = \frac{\phi \sin (l - \oplus)}{\cos \lambda}.$$

Latitude affected by parallax – true lat. $= ss' \cos Kss'$

$$= \phi \sin Zs \cos ZsL = \phi \sin Zs \tan Ls \cot Zs$$

$$= \phi \tan Ls \cos Ls \cos LZ = \phi \sin Ls \cos LZ$$

$$= \phi \sin \lambda \cos (l - \oplus).$$

The parallax in latitude can never exceed ϕ; and is, therefore, not so proper for attempting to detect ϕ from observation, as the parallax in longitude.

266. *To find, by observation, the annual parallax of a given fixed star, if it has any.*

Let the star's longitude be found when $l - \oplus = 90^0$, then if l be the longitude seen from the Sun, the apparent longitude

$$= l + \phi \,.\, \sec \lambda.$$

At the end of half a year, $\oplus - l = 90^0$ and the apparent longitude

$$= l - \phi \,.\, \sec \lambda;$$

therefore, if Δ be the observed difference of the star's longitudes at these times,

$$\Delta = 2\phi \,.\, \sec \lambda; \;\therefore\; \phi = \tfrac{1}{2} \,.\, \Delta \cos \lambda.$$

Secular Parallax.

267. All the fixed stars have slight *apparent* changes of place, arising from causes which are well known, and which will be explained in the Chapters on Precession and Aberration. Besides these, many of the stars have other changes of position, which are generally believed to be proper motions; though some writers, who considered that they were not independent of each other, have thought that they were only apparent, and caused by the motion of our system about some distant point, perhaps the center of gravity of the universe. The effects of such a motion of the Earth upon the apparent places

of the stars, will evidently be similar to that produced by parallax, and may be calculated by similar formulæ; but as the Earth's radius was considered evanescent in the investigations of the annual parallax, so in these motions, which, (if the hypothesis of their cause be correct) arise from a *secular* parallax, the whole of the solar system must be considered as occupying a mere point.

If AA' (fig. 31.) be the arc described in a century by our solar system, an immoveable star C will be seen in the directions AC, $A'C$, when the Earth is at A and A'; therefore considering AA' a straight line, on account of the immense radius of the orbit of which it forms a part, and making ACA' the secular parallax $= \pi$, $CAA' = m$, $AA' = r$, and $AC = \rho$,

$$\sin \pi = \frac{r}{\rho} \cdot \sin m.$$

In this equation r and ρ are unknown, and will probably remain so; therefore the absolute secular parallax cannot be discovered. We have it, however, in our power to determine by observation the *direction* of the motion of the solar system, if any such motion exists.

268. Let P (fig. 29.) be the pole of the equator, Z that point of the heavens towards which our system is supposed to move, S a fixed star, which, in consequence of the Earth's motion from the center of the sphere towards Z, appears to describe the small arc SS' in ZS produced; exactly in the same manner as the displacement of the observer from the Earth's center to the place whose zenith is Z, apparently depresses the star from S to S'; then, if x be the right ascension, and y the declination of the point Z, a and δ the right ascension and declination of S, m and n the apparent motions of S in right ascension and declination, and ϕ the motion which a star 90° from Z would appear to have; we have, as in Art. 249,

$$m = \phi \cdot \frac{\sin ZP \cdot \sin ZPS}{\sin PS} = \phi \cdot \frac{\cos y \cdot \sin (a - x)}{\cos \delta}, \ldots\ldots\ldots\ldots (1).$$

$$n = \phi \cdot \{\cos ZP \cdot \sin PS - \cos PS \cdot \sin ZP \cdot \cos ZPS\}$$

$$= \phi \, \{\sin y \cdot \cos \delta - \sin \delta \cdot \cos y \cdot \cos (a - x)\} \ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating ϕ between (1) and (2), and substituting for $\sin(a-x)$, $\cos(a-x)$, their expansions,

$$m\,.\,\{\sin y\cos\delta-\sin\delta\,.\cos y\,.\,(\cos x\,.\cos a+\sin x\,.\sin a)\}$$

$$=n\cos y\,.\,\frac{(\sin a\cos x-\cos a\sin x)}{\cos\delta};$$

therefore, multiplying by $\cos\delta$, and dividing by $\sin y$,

$$m\,\{\cos^2\delta-\sin\delta\cos\delta\cos a\,.\cot y\,.\cos x$$

$$-\sin\delta\,.\cos\delta\,.\sin a\cot y\sin x\};$$

$$=n\cos x\cot y\sin a-n\sin x\cot y\cos a;$$

$$\therefore\ m\cos^2\delta=(m\sin\delta\cos\delta\cos a+n\sin a)\cot y\,.\cos x$$

$$+\ (m\sin\delta\cos\delta\sin a-n\cos a)\,.\cot y\,.\sin x,$$

or, calling the known coefficients p and q,

$$m\cos^2\delta=p\,.\cot y\cos x+q\,.\cot y\sin x.$$

Another simple equation may be formed in the same manner from the apparent motions of another star, and from these two equations the values of $\cot y.\cos x$, $\cot y.\sin x$, and, therefore, of y and x may be determined.

269. When the numerical values deduced from observation are employed for determining y and x, the values which are deduced from the observation of different stars, differ considerably from each other, whence it may be concluded, that the apparent motion of the stars does not arise solely from the motion of the solar system.

CHAPTER X.

ON PRECESSION AND NUTATION.

270. THE comparison of ancient and modern observations shews that the longitude of every fixed star has an annual increase of about $50'',2$, whilst the latitude remains nearly unchanged. This increase, which arises from the regression of the first point of Aries from which the longitudes are reckoned, and not from a real motion of the stars, is called the *Precession of the Equinoxes.* The reason of the name as well as the cause and some of the consequences of Precession were explained in the Introduction.

The same observations point out a contemporary decrement of the obliquity, which is called the *Secular diminution of the Obliquity*, and amounts to $0'',48$ or about half a second yearly.

Both these uniform changes are subject to variations, which recur after a period of 18 years, and are called *Nutations*.

271. Precession and Nutation arise principally from the attractive forces of the Sun and Moon acting on the protuberant matter of the Earth's equator. These forces may be resolved into two, one in the plane of the equator which cannot change the position of that plane; and the other perpendicular to it, which, if the Earth had no rotation, would diminish the inclination of the planes of the ecliptic and equator, without altering the line of their intersection. But, in consequence of the Earth's rotation, these effects are reversed, and the inclination remains unaltered, whilst the line of equinoxes moves backwards on the ecliptic annually through an angle called the *Luni-solar Precession* the present value of which is $50'',3538$.

The disturbing forces however, although thus altering the direction of the Earth's axis in space, do not alter the velocity of the Earth's rotation, nor displace the poles of rotation on its surface; that such is the case appears both from the sea maintaining its level, which it could not do if the motion of the axis were not accompanied with a motion of the whole mass of the

Earth; and from the latitudes of places on the Earth having undergone no sensible change from the earliest times.

272. The planets are too distant to produce any sensible effect on the protuberant matter of the Earth, but by their mutual attractions they produce a slight change in the position of the ecliptic, causing the secular diminution of the obliquity, and ? an annual motion forwards of the equinoxial points through an angle whose present value is about 0″,12″; this being subtracted from the Luni-solar Precession, $50'',35$, leaves $50'',23$ for the present value of the annual regression of the Equinoxes, which is termed the *General Precession*. These effects of the planetary attractions are independent of the figure of the Earth, but as they change the plane of the Sun's motion, they must cause a corresponding change in the Sun's action on the protuberant matter of the Earth, similar to the lunar nutation, but much less in quantity, and immensely greater in its period.

273. As it thus appears that the positions of the Ecliptic and Equator are continually changing, it becomes necessary to have some fixed great circle of the sphere relative to which, at any assigned time, those positions may be determined. This after the example of Laplace, is generally taken to be the great circle with which the Ecliptic coincided at the beginning of the year 1750; let K (fig. 32.) be its pole, and K' the pole of the ecliptic in $1750 + t$; KK' their inclination $= p$, R any point in KK' produced; P the pole of the equator in 1750, P' the pole of the equator in $1750 + t$, so that at that time $P'K = \omega$ is the obliquity of the fixed ecliptic, and $P'K' = \omega'$ the obliquity of the actual ecliptic. Also let ♈ be the intersection of the ecliptic and equator in 1750, and ♈′♈″ the position of the equator in $1750 + t$, so that ♈″ is the actual equinox at that time, and ♈′ is the position it would have had, if there had been no motion of the pole of the ecliptic. Let the two positions of the ecliptic intersect on the line Nn; and let the longitude of the ascending node of the actual ecliptic upon the fixed ecliptic reckoned from the equinox of 1750 or ♈$N = \Pi$; the motion of the point of intersection of the equator with the fixed ecliptic, reckoned from the same epoch or ♈♈′$= \psi$, which is the luni-solar precession; and the motion of the points of intersection of the equator with the actual Ecliptic

or $N\text{♈}'' - N\text{♈} = \phi$, which is the general precession; and the motion of the ecliptic upon the equator or $\angle KP'K' = \lambda$. Then the values of the quantities ψ, ϕ, ω, ω', given by Physical Astronomy are

$$\psi = t\,(50'',37572) - t^2\,(0'',0001217945)$$

$$\phi = t\,(50'',21129) + t^2\,(0'',0001221483)$$

$$\omega = 23^0.\,28'.\,18'' + t^2\,(0'',00000984233)$$

$$\omega' = 23^0.\,28'.\,18'' - t\,(0'',48368) - t^2\,(0'',00000272295).$$

274. We shall now shew how the actual positions of the ecliptic and equator in the year 1750 + t may be determined relative to the great circle which we have assumed to be fixed. From the spherical triangle $KP'K'$, observing that

$$\angle K'KP' = K'KN - P'KN = 90^0 - (\text{♈}'KN - \text{♈}'KP') = 180^0 - \Pi - \psi$$

$$\angle KK'P' = 90^0 + RK'\text{♈}'' = 90^0 + (\Pi + \phi - 90^0) = \Pi + \phi,$$

we get

$$\tan \tfrac{1}{2} p \sin \left\{\Pi + \tfrac{1}{2}(\psi + \phi)\right\} = \sin \tfrac{1}{2}(\psi - \phi) \tan \tfrac{1}{2}(\omega + \omega'),$$

$$\tan \tfrac{1}{2} p \cos \left\{\Pi + \tfrac{1}{2}(\psi + \phi)\right\} = \cos \tfrac{1}{2}(\psi - \phi) \tan \tfrac{1}{2}(\omega' - \omega),$$

$$\tan \tfrac{1}{2} \lambda \cos \tfrac{1}{2}(\omega' + \omega) = \cos \tfrac{1}{2}(\omega' - \omega) \tan \tfrac{1}{2}(\psi - \phi),$$

from which, substituting the values of ψ, ϕ, ω, ω', and developing the results in series of powers of t, we find

$$\Pi = 171^0.\,36'\,10'' - t\,(5'',21)$$

$$p = t\,(0'',48892) - t^2\,(0'',0000030719),$$

$$\lambda = t\,(0'',17926) - t^2\,(0'',0002660394).$$

275. *Given the latitude and longitude of a fixed star at one epoch, to find them at any other.*

Retaining the same construction and notation, let S (fig. 32.) be a fixed star, SC (λ_1), its latitude at the fixed epoch 1750, SC' (λ) its latitude at any time $1750+t$, l_1 and l the corresponding longitudes.

$$\begin{aligned} \text{Then } l &= \text{♈}''N + NO + C'O \\ &= \text{♈}''N + NC + C'O \\ &= \text{♈}''N + l_1 - N\text{♈} + C'O \\ &= l_1 + \phi + C'O. \end{aligned}$$

But $C'O = \sin C'O = \tan\lambda \,.\, \cot SOC'$,

$$\text{and } \cot SOC' = \tan CNO \,.\, \cos NO = p \,.\, \cos(l_1 - \Pi)\,;$$

$$\therefore\ l = l_1 + \phi + p \,.\, \tan\lambda \,.\, \cos(l_1 - \Pi) \ldots\ldots (1),$$

$$\text{and } \lambda = SC - CO = \lambda_1 - p \,.\, \sin(l_1 - \Pi) \ldots\ldots (2).$$

Similarly, if l', λ' and p' be values of l, λ and p, for any other time $1750 + t'$,

$$l' = l_1 + \phi' + p' \,.\, \tan\lambda' \,.\, \cos(l_1 - \Pi') \ldots\ldots (3),$$

$$\lambda' = \lambda_1 - p' \,.\, \sin(l_1 - \Pi') \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

But $\tan\lambda' = \tan\lambda$ nearly; and $l_1 - \Pi$ and $l_1 - \Pi'$ each nearly equals $l_1 - \frac{1}{2} \,.\, (\Pi + \Pi')$; therefore calling this quantity L, and eliminating l_1 between (1) and (3), and λ_1 between (2) and (4), we get

$$l' = l + \phi' - \phi + (p' - p) \,.\, \tan\lambda \,.\, \cos L,$$

$$\text{and } \lambda' = \lambda - (p' - p) \,.\, \sin L\,;$$

therefore substituting in the equations the values of ϕ, p, &c. we get

$$\begin{aligned} l' = l &+ (t' - t) \,.\, \{50'',21129 + (t' + t)\, 0'',0001221483\} \\ &+ (t' - t) \,.\, \{0'',48892 - (t' + t)\, 0'',0000030719\} \,.\, \tan\lambda \,.\, \cos L, \end{aligned}$$

$$\lambda' = \lambda - (t' - t) \,.\, \{0'',48892 - (t' + t)\, 0'',0000030719\} \sin L,$$

$$\begin{aligned} \text{where } L &= l_1 - \tfrac{1}{2}(\Pi + \Pi') = l - \phi - \tfrac{1}{2}(\Pi + \Pi'), \text{ nearly} \\ &= l - 171^{\circ} \,.\, 36' \,.\, 10'' - (50'',21) \,.\, t + (5'',21)\,(t + t'). \end{aligned}$$

The above methods of finding the values of l' and λ' will not apply, unless p, which does not amount to 1′ in a century, is very small; and for very considerable intervals the relations between l' and l, or between λ' and λ, must be found by the common rules of Spherical Trigonometry.

276. From the latitudes, longitudes, and obliquity corrected as above, the right ascensions and declinations may be found; but it is obvious from the inspection of the values of l' and λ', that the latitude may be considered as unchanged for moderate intervals, whilst the longitude has an annual increase of $50'',2$; and it is upon this supposition that the precession in right ascension and declination for intervals not exceeding 5 years, is found in the next article.

277. *To find the precession in right ascension and declination of a given star.*

Let P (fig. 33.) be the pole of the equator, K of the ecliptic, ΥP the equinoctial colure, S and S' two apparent consecutive positions of a fixed star in the first quadrant, after an interval of t years; then the co-latitudes KS, KS' remain unaltered, and SKS', the apparent motion in longitude, $= (50'',2)\,t$.

Draw $S'O$ perpendicular to PS, and let

$$SO \text{ the precession in declination } = a,$$

$$SPS' \text{ the precession in right ascension } = \beta,$$

$$\text{and the angle of position } PSK = S.$$

$$\begin{aligned}\text{Then } a = SS' . \cos S'SO &= 50'',2t . \sin KS . \sin PSK\\ &= 50'',2t . \sin PK . \sin KPS\\ &= 50'',2t . \sin \omega . \cos a,\end{aligned}$$

$$\text{since } KPS = 90^0 + a.$$

Hence, Declination affected by precession – mean declination

$$= t . 50'',2 \sin \omega \cos a.$$

Again, in SP produced make $SQ = 90^0$ and join KQ by the arc of a great circle; then

$$\begin{aligned}\beta \sin PS = S'O = SS' . \sin S'SO &= 50'',2t . \sin KS \cos S\\ &= 50'',2t . \cos KQ, \text{ from the triangle } KSQ,\\ &= 50'',2t . (\cos KP \cos PQ + \sin KP \sin PQ \cos KPQ);\end{aligned}$$

$$\therefore \ \beta = 50'',2t (\cos \omega + \sin \omega \tan \delta \sin a),$$

$$\text{since } PQ = \delta, \ \angle KPQ = 90^0 - a.$$

Hence, R.A. affected by precession − mean R.A.

$$= t \,.\, 50'',2 \,(\cos\omega + \sin\omega \tan\delta \sin a).$$

The annual precession in declination and right ascension may be written $\Delta\delta = n\cos a$, $\Delta a = m + n\sin a \tan\delta$. For the year 1830, the values of the constants m and n were

$$m = 46'',02061, \ n = 20'',04263.$$

The quantities a and β are what must be added to the registered, to get the present values. If the angle of position $= 90^0$, or 270^0, the precession does not alter the right ascension.

On Nutation.

278. The expressions given in Art. 273, for ω' and ϕ, the quantities which determine the position of the equator relative to the actual ecliptic at the time $1750 + t$, contain only the terms which depend upon t, and furnish only the mean obliquity and the mean precession; but theory shews that the complete expressions for ϕ and ω' contain likewise the quantities $\Delta\phi$ and $\Delta\omega'$ made up of periodic terms depending upon the places of the Moon's node, and of the Sun and Moon; so that in the year $1800 + t$, the true longitude of a star, whose mean longitude in 1800 was l_0, will be

$$l_0 + t \,.\, (50'',2235) + t^2 (0,0001221805) + \Delta\phi,$$

and the inclination of the actual ecliptic and equator

$$23^0 . 27' . 54'',8 - t \,.\, (0'',457) + \Delta\omega;$$

where $\Delta\phi = -16'',78332 \sin \text{☊} - 1'',33589 \sin 2\odot$

$$+ 0'',20209 \sin 2\,\text{☊} - 0'',20128 \sin 2\,\text{☽};$$

$$\Delta\omega = +8'',97707 \cos \text{☊} + 0'',57990 \cos 2\odot$$

$$- 0'',08773 \cos 2\,\text{☊} + 0'',08738 \cos 2\,\text{☽};$$

☊ denoting the mean longitude of the Moon's ascending node, and ⊙ and ☽ the true longitudes of the Sun and Moon, respectively.

Since these expressions, $\Delta\phi$ and $\Delta\omega$ have their origin in a nutation of the Earth's axis, and consequently in a motion of the equator, the latitudes of the stars are not affected by them; but the longitudes, as they are reckoned from the equinox altered by nutation, all receive the common increment $\Delta\phi$. If the equations (1) and (2) (Art. 185.) be differentiated with respect to a, δ, l and ω, and for $\Delta l\ (=\Delta\phi)$ and $\Delta\omega$ the above values be substituted, the corresponding variations of a and δ, or the nutation in right ascension and declination, may be found. But for our present purpose it is more convenient to use the following process. The effect of the terms depending upon ☊, ⊙, and ☽, are called respectively the lunar inequality of precession, and the lunar nutation; the solar inequality of precession, and the solar nutation; and the monthly lunar inequality of precession, and monthly lunar nutation.

279. To find the nutation of a given star in right ascension and declination.

At any time $1800 + t$, let K (fig. 34.) be the pole of the ecliptic $♈L$, P the mean place of the pole of the mean equator $♈s$, and p the true place of the pole of the equator $♈'s'$. Then because the first term in the values of Δl and $\Delta\omega$ is by far the most considerable, if the Moon's ascending node should at this time coincide with the first point of Aries, p would be nearly in KP produced, more distant from K than the mean place by about $9''$; the actual place of p will be determined by taking $Kp = \omega + \Delta\omega$ on the circle of latitude Kl which makes with KPL an angle $pKP = \Delta l$. Therefore drawing po perpendicular to KP, we shall have $po = \Delta l \,.\, \sin\omega$, and $Po = \Delta\omega$, nearly. Let S be a star whose declination is δ, and right ascension a between 90^0 and 180^0; draw the declination circles PSs, pSs' through S, Ps meeting the equator $♈'s'$ in t, and draw $♈v$ perpendicular to the equator, and mon perpendicular to PS or pS. Then,

Declination affected by nutation − mean Declination.

$$= SP - Sp = Pm - pn = Po\cos(a - 90^0) - po\sin(a - 90^0)$$

$$= \Delta\omega \,.\, \sin a + \Delta l \,.\, \sin\omega\cos a.$$

R. A. affected by nutation − mean R. A.

$$= ♈'s' - ♈s = ♈'v + ts'.$$

$$\text{But } \Upsilon' v = \Upsilon\,\Upsilon' \cos\omega = \Delta l \cos\omega,$$

$$\begin{aligned} \text{and } ts' &= \tan\delta\,.\,mn = \tan\delta\,(mo + no) \\ &= \tan\delta\,(-Po\,.\cos a + po\,.\sin a) \\ &= \tan\delta\,(-\Delta\omega\,.\cos a + \Delta l\,.\sin\omega\sin a); \end{aligned}$$

$\therefore$ R. A. affected by nutation – mean R. A.

$$= \Delta l\,.\,(\cos\omega + \sin\omega\sin a\tan\delta) - \Delta\omega\,.\cos a\tan\delta.$$

Hence, substituting for Δl and $\Delta\omega$ their values, taking however only the most considerable terms depending upon ☊ and $2\odot$, namely,

$$\Delta l = -16'',783\sin ☊ - 1'',335\sin 2\odot = -(g\sin ☊ + g'\sin 2\odot)\operatorname{cosec}\omega,$$

$$\Delta\omega = +8'',977\cos ☊ + 0'',579\sin 2\odot = \quad h\cos ☊ + h'\sin 2\odot,$$

suppose, we get

Declination affected by nutation – mean Declination

$$= (h\cos ☊ + h'\cos 2\odot)\sin a - (g\sin ☊ + g'\cos 2\odot)\cos a,$$

R.A. affected by nutation – mean R.A.

$$\begin{aligned} &= -(h\cos ☊ + h'\cos 2\odot)\cos a\tan\delta \\ &\quad - (g\sin ☊ + g'\sin 2\odot)(\cot\omega + \sin a\tan\delta). \end{aligned}$$

280. The change of the equinoxial point in longitude $\Upsilon\,\Upsilon'$

$$= -16'',783\sin ☊ \text{ nearly};$$

and the change in R.A. $= \Upsilon\,\Upsilon'\cos\omega = -16'',783\sin ☊\cos\omega$, which is called the equation of the equinoxes in R.A.

CHAPTER XI.

ON THE ABERRATION OF LIGHT.

281. The last of the corrections to be considered is Aberration. If light be not propagated instantaneously, and if its velocity bear any assignable ratio to the velocity of the spectator resulting from the Earth's motion in her orbit, the direction in which rays from a heavenly body will enter the eye from the combined effect of these motions, and in which the body will appear, will be different from the line joining the body and the eye at the same instant which is its true direction; the angle between these directions is called *aberration*.

That light is not transmitted instantaneously was shewn by Roëmer, who observed that the eclipses of Jupiter's first satellite happened sooner than they ought by computation when Jupiter was in opposition, and therefore nearer the Earth than when at his mean distance, and later when Jupiter was in conjunction and therefore more remote; and that these irregularities were attributable to the unequal times which light from the satellite employed to pass to the spectator from Jupiter at his different distances. From a great variety of observations, it appears that light comes from the Sun at his mean distance of about 96 millions of miles in $8^{m}. 13^{s}$; hence the velocity of light is more than ten thousand times greater than the velocity of the Earth which is 19 miles in a second.

282. To understand how the velocity of the Earth in her orbit, combined with the velocity of light, makes the apparent place of a star differ from its true one, the following illustrations may be of assistance.

Suppose *SA* (fig. 35.) the open window of a carriage at rest; then drops of rain falling in a direction parallel to *SA*, will none of them enter the window; but if, whilst a drop falls through *SA*, the carriage move over *BA*, the drop being caught when at *S* before it reaches the ground will fall on the point B within the carriage, and the traveller not attending to his own motion, will suppose the rain to beat in, in the direction SB or sA, As being parallel to BS. The motion of light combined with that of the Earth produces a similar deception with respect to the places of the fixed stars, so that if the Earth in its orbit move over BA, whilst the light from a fixed star moves over SA, the light will appear to come in the direction sA, or the star will be seen in the direction As, instead of the true direction AS. The angle SAs, which measures the apparent deviation of the star from its true place, is the *Aberration.*

283. Again, let S (fig. 16. Plate I.) be a star, T a spectator at the Earth's center, and Tt a portion of a tangent to the Earth's orbit at T described by the Earth in the time light takes to pass from S to T; for since the velocity of the Earth's rotation, being sixty times less, may be neglected in comparison with the velocity of translation, and since rays from every point of the disk of a heavenly body may be supposed to be parallel to the line joining its center with the Earth's center, the phenomena under consideration will be presented to an observer any where on the surface, just the same as if he were at the center. To S and T, suppose velocities equal and opposite to the Earth's velocity communicated which will not alter their positions relative to the line Tt; then the spectator will be at rest; and drawing Ss parallel and equal to Tt, the star will be at s, at the instant the ray it emitted when at S reaches T, i. e. the apparent direction TS corresponds to the real direction Ts; and drawing tS' parallel to TS, when the spectator is at t, the star will be seen in the direction tS'.

284. Besides, it is manifest that if TS were a tube carried parallel to itself along with the Earth, a ray entering it at S in the direction ST would, by impinging against its side, be immediately stopped; but a ray entering in the direction St would always be in the axis; for when the tube was in the position mn, we should have $Sa : ab :: St : tT ::$ vel. of light : vel. of Earth, therefore the ray would be at a when the Earth was at m; and when the ray entered the eye at t, the tube would have the position tS', and consequently the body would appear in that direction.

Hence the aberration takes place in a plane passing through the heavenly body, the Earth, and a tangent to the Earth's orbit, and is always in the direction towards which the Earth is moving, i. e. to the point of the ecliptic 90° behind the Sun's place; the heavenly body is consequently elevated or depressed with reference to the line of the spectator's motion, according as he is moving from or towards the body.

285. The phenomenon of aberration, though not the most striking, is one of the most pleasing in Astronomy, and its explanation, though refined, is perfectly satisfactory. It affords a direct proof of the motion of the Earth in her orbit; and the exact correspondence of the velocity of light deduced from the observed aberrations of the fixed stars, with that given by the eclipses of Jupiter's first satellite, as also of the apparent places of the stars with those deduced from the formulæ of aberration, leaves not a doubt of the truth of the theory. In consequence of aberration none of the heavenly bodies are in the places in which they seem to be; they are made visible by rays which left them some time previously, during which we have changed our position, and the eye strikes the light in an altered direction; consequently we refer them to situations in which they are not. Thus the center of the Sun appears always $20''$ less advanced in the ecliptic than it would be, if light came instantaneously; also at Sun-rise we receive the first rays $8^m.13^s$ after their emission, and the last rays at the same interval after Sunset. The satellites of Jupiter have, for some minutes, passed from their eclipse before we are sensible of it; and the alterations in the positions of those stars which have small proper motions may precede by several years the instant of their being observed here. The places of the Moon and Planets, by reason of their proper motions, are affected by Aberration in a somewhat different manner from those of the fixed stars; the amount is estimated first by considering the effect of the Earth's motion on the apparent place, and then the aberration arising from the body's own motion.

286. *To find the aberration of a fixed star.*

Let ⊙ be the Sun, (fig. 35. bis) T the Earth in its orbit, ♈ LE that orbit extended to the fixed stars, or the ecliptic;

TO a tangent to the Earth's orbit at T, S a fixed star in the circle of latitude KSL, SO the intersection of a plane through OT and S with the celestial vault. Let Tt be a portion of the orbit traversed by the Earth's center, in which we suppose the spectator placed, whilst light describes ST, that is, let ST, Tt be to one another in the proportion of the velocities of light and the Earth; complete the parallelogram Ts, then Ts is the apparent direction of the star, and STs or Tst the aberration; but

$$\frac{\sin Tst}{\sin sTt} = \frac{Tt}{st} = \frac{\text{vel. of Earth}}{\text{vel. of light}};$$

and $sTt = STO$ nearly, = the angle which a line joining the Earth and star makes with the direction of the Earth's motion, which angle is called the *Earth's way;* also $\sin Tst = Tst = STs$ nearly;

$$\therefore \text{ aberration} = \sin \oplus\text{'s way} \times \frac{\text{vel. of } \oplus}{\text{vel. of light}}.$$

Now light is $8^{m}.13^{s}$ in passing over a radius of the Earth's orbit (r), and in this time the Earth describes an angle of $20'',5$ round the Sun, or an arc $= 20,5 \sin 1'' r$;

$$\therefore \text{ the aberration} = \frac{20,5 \sin 1'' r}{r} \sin \oplus\text{'s way} = 20'',5 . \sin \oplus\text{'s way}.$$

287. It is most convenient, in treating of the effects of aberration on the apparent places of the stars, to refer them to the surface of a sphere having the Sun in its center. As the plane STO, in which the aberration takes place, does not pass through the Sun, its intersection SO with the surface of the sphere is not a great circle; but if we suppose a plane to pass through the Sun and the true and apparent places of a fixed star, this plane may be considered parallel to the former, on account of the immense distance of the stars; and the aberration and Earth's way may be considered without sensible error, as measured on the arcs of the great circle in which this plane cuts the surface of the sphere. The plane thus drawn being parallel to the direction of the Earth's motion, cuts the ecliptic in two points 90° distant from E the Earth's place, if the Earth's orbit be supposed circular. In order to represent then the effect

of aberration on a given star at a given epoch, take E (fig. 38.) the corresponding place of the Earth referred to the ecliptic, make $EO = 90^0$, and through S the fixed star draw the great circle OS, and on it measure $Ss = 20'',25 \sin OS$ in the direction OS, because we suppose the Earth to move in the order ♈ OE; then s is the apparent place of the star.

288. If S (fig. 35. bis) be the true, and s the apparent place of a fixed star, Ss is parallel to the direction in which the Earth moves, and therefore, to the ecliptic, and is to ST the distance of the star from the Earth as vel. Earth to vel. light. The velocity of light is constant, and that of the Earth may be supposed uniform, since the Earth's orbit, as will be shewn in the next Chapter, is nearly circular; also TS may be considered constant and parallel to itself in all positions of the Earth, because the diameter of the Earth's orbit subtends no finite angle at S; therefore Ss is constant, and the apparent place s describes a circle round S, the true place, in a plane parallel to the ecliptic, in the course of a year. This circle is seen from the Earth, projected on a plane perpendicular to TS, or to $\odot S$, and will therefore appear elliptical, unless TS be perpendicular to its plane, or S be situated in the pole of the ecliptic. But the inclination of $S\odot$ to the plane of the circle described by s round S, or to the plane of the ecliptic, which is the same thing, is SL the star's latitude; therefore the $\frac{1}{2}$ axes of the projection of that circle upon a plane perpendicular to OS will be Ss, and $Ss \sin SL$; therefore, the major axis of the ellipse which a star appears to describe about its real place, in consequence of aberration, $= 41''$,

and the minor axis $= 41''$. sin star's latitude.

289. Hence it results that the stars will seem to have small annual proper motions, corresponding exactly to what was observed by Bradley when, in 1725, in endeavouring to determine the parallax of γ Draconis he discovered the aberration of light. A star in the pole of the ecliptic appears to describe a small circle parallel to the plane of the ecliptic about that point, whose diameter is $41''$; stars situated in the ecliptic appear to describe in that plane $20'',5$ on each side of their true places in the course of a year; stars in

intermediate situations seem to describe ellipses whose centers are their true places, greater axes parallel to the ecliptic and 41″ in length, and lesser axes proportional to the sine of the star's latitude.

290. *To find the resolved part of the aberration perpendicular to a given plane.*

Let E be the Earth's place in the ecliptic, (fig. 38.) S the true place of a fixed star; take $OE = 90^0$, then the great circle OS is the plane in which the aberration Ss takes place, s the apparent place of the star being deflected from O, since ♈ OE is the direction of the Earth's motion, Art. 287. Let IS be the given plane passing through the star, the deflection from which caused by aberration is to be estimated, I being the point of intersection of the given plane and the ecliptic, *behind* E; draw $s\sigma$ perpendicular to IS, then

$$s\sigma = Ss \,.\, \sin OSI = 20'',5 \,.\, \sin OS \,.\, \sin OSI$$

$$= 20'',5 \sin I \sin OI = 20'',5 \sin I \cos IE.$$

Hence the aberration perpendicular to IS ($s\sigma$) towards E $= 20'',5 \sin I \cos IE$; when $\cos IE$ is negative, the aberration towards E will be negative, i. e. it will be from E.

291. *To find the aberration of a given star in longitude.*

Let E be the Earth's place in the ecliptic (fig. 39.), and consequently H the Sun's place as seen from the Earth; S the true place of a star whose latitude and longitude are respectively λ and l, s the apparent place. Through K the pole of the ecliptic draw the circles of latitude KS, Ks, meeting the ecliptic in I, i; then

$$\text{aberration in longitude} = \Upsilon i - \Upsilon I = Ii = \frac{s\sigma}{\sin Ks}$$

$$= \frac{\text{aberration perpendicular to } IK}{\sin KS} \text{ nearly}$$

$$= \frac{20'',5 \sin I \cos IE}{\sin KS} = 20'',5 \,.\, \frac{\cos(180^0 + \odot - l)}{\cos\lambda},$$

or apparent long. − true long. $= -20'',5\,\dfrac{\cos(\odot - l)}{\cos\lambda}$;

$\odot$ being the Sun's longitude seen from the Earth $♈ H$, and

therefore $IE = ♈ E - ♈ I = 180^0 + \odot - l$.

292. *To find the aberration of a given star in latitude.*

Draw the great circle $I'S\sigma'$ (fig. 39.) perpendicular to KI or Ki, then aberration in latitude $= is - IS$

$= s\sigma'$ = aberration perpendicular to $I'S$,

$= -20'',5 \,.\, \sin I' \,.\, \cos I'E$.

But since the angles $I'SI$, $I'IS$, are both right angles, I' is the pole of KSI, and $\therefore \angle I' = IS = \lambda$;

also $I'E = 180^0 + \odot - (l - 90^0) = 270^0 + \odot - l$;

$\therefore$ apparent latitude − true latitude $= -20'',5 \sin\lambda \sin(\odot - l)$,

293. *To find the aberration of a given star in right ascension.*

Let $♈ E$ be the ecliptic (fig. 40.), $♈ A$ the equator, K and P their poles, S a fixed star, E the Earth's place, $EO = 90^0$, OS the great circle in which the aberration Ss takes place, and which tends from O because the Earth's motion is in that direction. Draw the declination circles PA, Pa through the true and apparent place of the star, and $s\sigma$ perpendicular to PS or Ps; then

aberration in right ascension $= ♈ a - ♈ A = Aa = \dfrac{s\sigma}{\sin Ps}$

$$= \frac{s\sigma}{\sin PS}, \text{ nearly, } = \frac{20'',5 \sin I \cos IE}{\cos\delta}.$$

Let $♈ I = \phi$, $♈ A = a$, then from the triangle $♈ IA$

$\cot\phi = \cos\omega \cot a$ which determines ϕ, and $\sin a = \sin\phi \,.\, \sin I$;

also $\cos IE = \cos(180^0 + \odot - \phi) = -\cos(\odot - \phi)$;

$$\therefore \text{ apparent R.A.} - \text{true R.A.} = -\frac{20'',5 \sin a \cos(\odot - \phi)}{\sin\phi\cos\delta}.$$

If $\odot - \phi = 90^0$ or 270^0, i. e. if the Sun's longitude

$$= 90^0 + \phi, \text{ or } 270^0 + \phi,$$

the aberration in right ascension vanishes.

294. *To find the aberration of a given star in declination.*

Draw the great circle $I'S\sigma'$ perpendicular to PA or Pa, (fig. 40.) meeting the equator in M, and ecliptic in I'; then aberration in declination $= as - AS = s\sigma'$ = aberration perpendicular to $I'S$

$$= -20'',5 . \sin I' \cos EI'.$$

Now M is the pole of AS, and therefore $\angle AMS = AS = \delta$; hence in the triangle $I' \Upsilon M$, we have

$$M\Upsilon = 90^0 - a, \quad M\Upsilon I' = \omega, \text{ and } I'M\Upsilon = 180^0 - \delta,$$

from which data $I'\Upsilon = \theta$ may be found by the usual methods. Also in the same triangle

$$\sin\theta \sin I' = \sin M\Upsilon \sin M = \cos a \sin\delta;$$

$\therefore$ apparent declination − true declination

$$= -20'',5 \frac{\cos a \sin\delta}{\sin\theta} \cos(180^0 + \odot + \theta)$$

$$= 20'',5 . \frac{\cos a \sin\delta \cos(\odot + \theta)}{\sin\theta}.$$

If the Sun's longitude $= 90^0 - \theta$ or $270^0 - \theta$, the aberration in declination vanishes.

295. In the four cases for which the aberration has been deduced, the Earth has been placed in such a position that the latitudes, longitudes, &c. have been increased by the effects of aberration; therefore the aberrations, calculated by these formulæ, must be all *added*, with their proper signs, to the true latitudes, longitudes, &c. to obtain the apparent values of the

same quantities. In the second and fourth cases, EI has been taken greater than 90°. These observations require attention in order to obtain the right sign of the aberration.

The subsidiary angles ϕ and θ have been introduced in the two last articles to facilitate the logarithmic calculation of the aberrations; there is, however, not much difficulty in obtaining formulæ depending on the right ascensions and declinations only, and these are most frequently employed.

296. *To find the aberration in right ascension and declination, in terms of* a *and* δ.

The aberration in right ascension $= \dfrac{s\sigma}{\cos\delta}$ (fig. 40.)

$$= 20'',5 \,.\, \frac{\sin I \,.\, \cos IE}{\cos\delta}$$

$$= \frac{20'',5}{\cos\delta} \,.\, \sin I \ \cos(\Upsilon E - \Upsilon I)$$

$$= \frac{20'',5}{\cos\delta} \,.\, \left\{\cos\Upsilon E \,.\, \sin I \,.\, \cos\Upsilon I + \sin\Upsilon E \,.\, \sin I \,.\, \sin\Upsilon I\right\};$$

$$\text{but } \sin I \,.\, \sin\Upsilon I = \sin a;$$

$$\therefore\ \sin I \,.\, \cos\Upsilon I = \sin I \,.\, \sin\Upsilon I \,.\, \cot\Upsilon I$$

$$= \sin a \,.\, \cos\omega \,.\, \cot a = \cos\omega \,.\, \cos a,$$

$$\text{and } \Upsilon E = 180^\circ + \odot;$$

therefore, apparent R. A. – true R. A.

$$= \frac{20'',5}{\cos\delta} \,.\, \left\{-\cos\odot \,.\, \cos\omega \,.\, \cos a - \sin\odot \,.\, \sin a\right\}$$

$$= -20'',5 \left\{\cos a \,.\, \cos\omega \,.\, \cos\odot + \sin a \,.\, \sin\odot\right\}\sec\delta.$$

The aberration in declination $s\sigma'$

$$= -20'',5 \,.\, \sin I' \,.\, \cos EI'$$

$$= -20'',5 \,.\, \sin I' \,.\, \cos(I'\Upsilon + \Upsilon E)$$

$$= -20'',5 \,.\, \left\{\cos\Upsilon E \,.\, \sin I' \,.\, \cos I'\Upsilon - \sin\Upsilon E \,.\, \sin I' \,.\, \sin I'\Upsilon\right\};$$

but $\sin I' . \sin I'\Upsilon = \sin\delta . \cos a$;

$\therefore\ \sin I' . \cos I'\Upsilon = \sin I' . \sin I'\Upsilon\ . \cot I'\Upsilon = \sin\delta \cos a \cot\theta.$

Also in the triangle $I'\Upsilon M$,

$$\sin a . \cos\omega = \cot\theta . \cos a - \sin\omega . \cot(180^0 - \delta);$$

$$\therefore\ \cot\theta = \tan a . \cos\omega - \frac{\sin\omega . \cot\delta}{\cos a};$$

$\therefore\ \sin I' . \cos I'\Upsilon = \sin\delta . \sin a \cos\omega - \cos\delta \sin\omega$;

therefore apparent declination $-$ true declination

$$= -20'',5\left\{-\cos\odot . (\sin\delta . \sin a \cos\omega - \cos\delta \sin\omega) + \sin\odot . \sin\delta . \cos a\right\}$$

$$= -20'',5(\cos a \sin\odot - \cos\omega \sin a \cos\odot)\sin\delta$$
$$-20'',5 \sin\omega \cos\odot \cos\delta.$$

297. *To find the aberration, taking into account the elliptic form of the Earth's orbit.*

At any point of the orbit we may suppose the Earth's velocity made up of two parts, one perpendicular to, and the other in the direction of the radius vector; call these v, v', and let any radius vector be produced to meet the ecliptic in E, (fig. 41.) take $EO = 90^0$, and through the fixed star S draw the great circle OSs; make $Ss = \frac{v}{V}\sin OS$, where V = velocity of light, then Ss will be the aberration arising from that part of the Earth's motion which is perpendicular to the radius vector. Also it is manifest that the effect of the motion in the direction KE will be precisely similar to that produced by the motion perpendicular to the radius vector, when the Earth's place was O; i. e. drawing the great circle ESO', and making

$$Ss' = \frac{v'}{V}\sin O'S,$$

Ss' will be the aberration in consequence of the motion in the direction of the radius vector.

Now, employing the usual notation, $v = \rho d_t\theta = \frac{h}{\rho}$; but on the supposition of the Earth's describing a circle whose radius is the mean distance, we have found

$$\text{its velocity } \left(\frac{h}{b}\right) = 20''\tfrac{1}{2}\,.\ \text{velocity of light};$$

therefore, eliminating h;

$$\therefore\ \frac{v}{V} = \frac{b}{\rho}\,.\,20''\tfrac{1}{2} = 20''\tfrac{1}{2}\,(1 + e\cos\theta) \text{ nearly, neglecting } e^2;$$

$$\text{also } v' = d_t\rho = \frac{1}{\rho}\,d_\theta\rho\,(\rho\,d_t\theta) = \frac{ve\sin\theta}{1 + e\cos\theta};$$

$$\therefore\ \frac{v'}{V} = 20''\tfrac{1}{2}\,e\sin\theta.$$

Let IS be any plane perpendicular to which the aberration is to be estimated; draw $s\sigma$, $s'\sigma'$ perpendicular to IS;

$$\therefore\ s\sigma = 20''\tfrac{1}{2}\,.\,\sin I\,.\,\cos IE\,(1 + e\cos\theta),$$

$$s'\sigma' = 20''\tfrac{1}{2}\,.\,\sin I\,.\,\cos IO\,.\,e\sin\theta;$$

$$\begin{aligned}\therefore\ \text{whole aberration perpendicular to } IS &= 20''\tfrac{1}{2}\,.\,\sin I\cos IE \\ &\quad + 20''\tfrac{1}{2}\,e\sin I\,(\cos IE\cos\theta + \sin IE\sin\theta) \\ &= 20''\tfrac{1}{2}\sin I\cos IE + 20''\tfrac{1}{2}\,e\sin I\cos(IE - \theta),\end{aligned}$$

$$\begin{aligned}\text{where } \theta &= \text{longitude of } \oplus - \text{longitude of perihelion} \\ &= (180^0 + \odot) - (180^0 + \Pi) = \odot - \Pi,\end{aligned}$$

Π being longitude of perigee.

The aberration caused by the eccentricity

$$= 20'',5e\,.\,\sin I\,.\,\cos(IE + \Pi - \odot),$$

which cannot exceed $20'',5e$ or $0'',25$, and may therefore be neglected. If, however, it should be thought useful to have the aberrations caused by the eccentricity, they may be easily obtained from the preceding values of I and IE. The aberrations thus deduced by Delambre, (*Astronomie Lecon.* XIX.) are

aberration in longitude

$$= -\,0'',34\,.\cos(\Pi - l)\,.\,\mathrm{sect}\,\lambda,$$

$$\text{aberration in latitude} = 0'',34\,.\,\sin\lambda\,.\,\sin(\Pi - \lambda),$$

aberration in right ascension

$$= -\,0'',34\,.\,(\cos\omega\,.\,\cos a\,.\,\cos\Pi + \sin a\,.\,\sin\Pi)\,.\,\sec\delta,$$

aberration in declination

$$= 0'',34\,.\,\left\{\sin\delta\,.\,(\cos\omega\,.\,\sin a\,.\,\cos\Pi - \cos a\,.\,\sin\Pi)\right.$$

$$\left. - \cos\delta\,.\,\sin\omega\,.\,\cos\Pi\right\}.$$

298. *To find the coefficient of aberration from observations of the stars without previous knowledge of the velocity of light.*

Let ♈ *LM* be the ecliptic, (fig. 42.) ♈ ♎, *LM*, the equinoctial and solstitial colures. At the vernal equinox when the Earth's place in the ecliptic is ♎, and the direction of its motion ♎ *M*, a star *S* in the solstitial colure, by the effect of aberration, is depressed to *s*, and $Ss = x \sin MS$; at the autumnal equinox when the Earth is at ♈ and moving towards *L*, the star is elevated to *s'*, and $Ss' = x \sin MS$; if therefore the difference of the latitudes or declinations at these times be observed = *D*, we have

$$D = ss' = 2Ss = 2x \sin MS = 2x \sin(\delta + \omega);$$

$$\therefore\ x = \frac{D}{2\sin(\delta + \omega)}.$$

Hence the polar distance of a star in the solstitial colure appears greatest in spring and least in autumn; also since the star's right ascension is 270°, it is 90° of right ascension west of the Sun in the former, and 90° east in the latter case; therefore it will cross the meridian of any place, most southerly at six in the morning in spring, and most northerly at six in the evening in autumn. The above is the explanation of the phenomena described by Bradley, as having been observed in the motions of γ Draconis, a star situated nearly as above, which led him to the discovery of the aberration of light.

299. The tables of aberration may be made use of in calculating the values of the annual parallax of the fixed stars, as the formulæ by which both these quantities are expressed, have a close analogy.

By reference to fig. 35, bis, it is seen that the annual parallax takes place in the plane STE, and varies as sin ES, and the aberration in the plane STO, and varies as sin SO; but since $\angle OTE = 90^0$, after three months the plane STE will be brought into the position STO, and the arc ES will become OS; consequently the formulæ which express the variation of aberration three months after the present time, represent the existing variations of annual parallax. If therefore in the formulæ for aberration we alter the coefficients, and increase the Sun's longitude by 90^0, we obtain the formulæ for annual parallax.

$$\text{Thus, aberration in longitude} = -20'',5 \frac{\cos(\odot - l)}{\cos\lambda},$$

$$\text{.................. latitude} = -20'',5 \sin\lambda \sin(\odot - l);$$

therefore, changing $\odot$ into $90^0 + \odot$, and replacing $20'',5$ by ϕ,

$$\text{annual parallax in longitude} = +\phi \frac{\sin(\odot - l)}{\cos\lambda},$$

$$\text{.................... latitude} = -\phi \sin\lambda \cos(\odot - l);$$

or if α and β represent the aberrations taken out of the tables corresponding to the Sun's longitude $90^0 + \odot$,

$$\frac{\phi}{20'',5} \cdot \alpha, \text{ and } \frac{\phi}{20'',5} \beta$$

will represent the parallaxes corresponding to the longitude $\odot$.

The annual parallax in right ascension and declination may be deduced in a similar manner from the aberration in right ascension and declination.

From the comparison of these expressions for the aberration and parallax, it appears that the parallax is greatest when the corresponding aberration is least, and conversely.

300. The diurnal motion of the Earth combined with the motion of light will also cause a small change in the places of the stars, but too small to have any observable influence, except perhaps on the right ascension of stars near the pole. The coefficient of this aberration for a place whose latitude is l

$$= \frac{\text{vel. of the place}}{\text{vel. of light}} = \frac{\text{vel. of equator} \,.\, \cos l}{\text{vel. of light}} = 0'',3 \cos l, \text{ nearly.}$$

301. *To find the aberration in right ascension and declination caused by the diurnal rotation.*

Let OE (fig. 43.) be the equator, P its pole, Z the zenith, draw the declination circle PZE, then the direction of the motion of the place is manifestly parallel to the tangent at E; take therefore $OE = 90^\circ$, and draw the great circle OS through the star S, then $Ss = 0'',3 \cos l \sin SO$,

$$\text{and aberration in right ascension} = \Upsilon i - \Upsilon I = Ii$$

$$= \frac{s\sigma}{\sin Ps} = \frac{s\sigma}{\cos \delta} \text{ (nearly)} = \frac{0'',3 \cos l \sin I \cos IE}{\cos \delta}$$

$$= \frac{0'',3 \cos l \cos h}{\cos \delta},$$

since $IE = \angle ZPS = h$, the star's hour angle.

Again, draw the great circle $I'S\sigma'$ perpendicular to PS, then aberration in declination $= s\sigma' =$ aberration perpendicular to $I'S$ $\qquad = -\,0'',3 \cos l \sin I' \cos I'E$,

$$\text{but } \angle I' = IS = \delta, \text{ and } I'E = 90^\circ + h;$$

$$\therefore \text{ aberration in declination} = 0'',3 \cos l \,.\, \sin \delta \,.\, \sin h.$$

These formulæ shew that at the superior transit a star's R. A. is increased, on account of the diurnal aberration, by the quantity $0'',3 \cos l \sec \delta$, and diminished by the same quantity at the inferior transit; and that the declination is unaffected at both transits.

302. *To find the effect of aberration on the Sun, Moon, and planets.*

Let p and e (fig. 44.) be cotemporary positions of the Earth and a planet, P and E other cotemporary positions after an interval (t^s) in which light moves from p to e. Then if the Earth were at rest at E, the planet would be seen by the ray pE, emitted when the planet was at p. Take now $Ee' = Ee$, and complete the parallelogram Eq, then pEq is the aberration caused by the Earth's motion; therefore the whole aberration $= PEq$, or the planet, when at P, will be seen in the direction Eq. But $PEq = PEp + pEq = PEp + Epe =$ the motion of the planet round E at rest + the motion of E round p at rest = the whole geocentric motion of the planet in t^s, which, if m be the geocentric motion of the planet in 1^s, $= mt$. Now if D be the planet's distance from the Earth, considering the distance of the Earth from the Sun unity, $t = D\,(8^m.13^s)$;

$$\text{therefore the aberration} = 493''\, mD.$$

This aberration may be resolved in any direction, and will preserve the same ratio with the planet's geocentric motion resolved in the same directions, so that

$$T = A - 493''\, mD,$$

where T is the true and A the apparent latitude, longitude, right ascension or declination of P, and m the increment of the same quantities in 1^s.

Fig. 44. represents the planet's motion retrograde, if the motion be direct as in fig. 45. $PEq = PEp - Epe =$, as before, the geocentric motion of P round E in t^s. In the case of the Sun, the true position is always in advance of the apparent, so that when the Sun's radius vector in parts of his mean dist. is ρ, true long. = apparent long. + $20'',5\rho$.

General Formulæ for the reduction of the Fixed Stars.

303. By collecting the results obtained in this and the preceding Chapter, we see that if t denote the time from the beginning of the year expressed in fractional parts of a year, $a =$ mean right ascension, and $\delta =$ mean declination for the beginning of the year; then for the time t,

apparent right ascension a'

$$= a - 20'',5\,(\cos\omega\cos\odot\cos a + \sin\odot\sin a)\sec\delta$$

$$+\, t\,(m + n\sin a\tan\delta) - (g\sin\text{☊} + g'\sin 2\odot)\,(\cot\omega + \sin a\tan\delta)$$

$$-\,(h\cos\text{☊} + h'\cos 2\odot)\cos a\tan\delta.$$

Apparent declination δ'

$$= \delta - 20'',5\,(\sin\odot\cos a - \cos\omega\cos\odot\sin a)\sin\delta$$

$$-\,20'',5\sin\omega\cos\odot\cos\delta + tn\cos a$$

$$+\,(h\cos\text{☊} + h'\cos 2\odot)\sin a - (g\sin\text{☊} + g'\cos 2\odot)\cos a.$$

Now assume

$$\begin{aligned}
A &= -20'',5\cos\omega\cos\odot,\\
B &= -20'',5\sin\odot,\\
C &= t - g\cot\omega\sin\text{☊} - g'\cot\omega\sin 2\odot,\\
D &= -h\cos\text{☊} - h'\cos 2\odot,
\end{aligned}$$

$$\begin{array}{ll}
a = \cos a\sec\delta, & a' = \tan\omega\cos\delta - \sin a\sin\delta,\\
b = \sin a\sec\delta, & b' = \cos a\sin\delta,\\
c = m + n\sin a\tan\delta, & c' = n\cos a,\\
d = \cos a\tan\delta, & d' = -\sin a.
\end{array}$$

Then we have the total correction for aberration, precession and nutation expressed by

$$\left.\begin{aligned}
\Delta a &= Aa + Bb + Cc + Dd\\
\Delta\delta &= Aa' + Bb' + Cc' + Dd'
\end{aligned}\right\}\ldots\ldots\ldots\ldots(1).$$ so that

apparent R.A. at time t = mean R.A. at beginning of year + Δa,
apparent Dec. at time t = mean Dec. at beginning of year + $\Delta\delta$.

The factors A, B, C, D are common to all stars, as they depend only on the time and the longitude of the Sun, and of the Moon's ascending node; a, b, c, d, a', b', c', d', are factors depending on each particular star. In the Nautical Almanac the logarithms of the independent factors A, B, C, D are computed for mean midnight at Greenwich throughout the year; the logarithms of the remaining factors depending on the star's place are given for 2881 principal stars in the Astronomical Society's Tables; by means of which logarithms, each of the products in (1) can be readily calculated, and taking the sum of these products the reduction of any star in the Astronomical Society's Catalogue can be effected with great ease.

For stars near the pole, since for them $\tan\delta$ becomes very large, it becomes necessary to take into account the terms in $\Delta\omega$ and Δl (Art. 278), which depend on $2☊$ and $2☽$; therefore for such stars the coefficients C and D must be increased by the terms

$$+0.00413\sin 2☊ - 0.004\sin 2☽\text{, and} + 0.09030\cos 2☊ - 0.090\cos 2☽,$$

respectively. Also for stars having proper motions, if a, a', be the annual proper motions in R.A. and declination respectively, Δa and $\Delta\delta$ must be increased by ta and ta'.

304. The formulæ (1) may be likewise modified so that the independent coefficients A, B, C, D may be made useful in the reduction of a star not in the Catalogue and for which, consequently, the coefficients a, b, c, d have not been computed. For, substituting for a, b, c, d their values in (1), we get

$$\Delta a = (A\cos a + B\sin a)\sec\delta + (Cn\sin a + D\cos a)\tan\delta + Cm,$$

$$\Delta\delta = A\tan\omega\cos\delta + (-A\sin a + B\cos a)\sin\delta + Cn\cos a - D\sin a.$$

Assume $\dfrac{A}{B} = \tan H$, $\dfrac{D}{Cn} = \tan G$;

$$\therefore \Delta a = B\sec H\sin(H+a)\sec\delta + Cn\sec G\sin(G+a)\tan\delta + Cm,$$

$$\Delta\delta = A\tan\omega\cos\delta + B\sec H\cos(H+a)\sin\delta + Cn\sec G\cos(G+a).$$

In the Nautical Almanac, the quantities Cm, $Cn\sec G$, G, $B\sec H$, H, and $A\tan\omega$, are computed for every 5th day in the year; and by means of their values, the above formulæ may be calculated with great facility.

CHAPTER XII.

THE THEORY OF ELLIPTIC MOTION.

305. THE path of the Earth round the Sun, or (which has been shewn to be the same thing) the apparent path of the Sun round the Earth, lies in the plane of the ecliptic*. The position of this plane in space is sufficiently determined by the latitudes of known fixed stars; and has been determined with reference to the equator, by the obliquity, and place of the equinox.

If the Sun's path in the ecliptic were a circle having the Earth in its center, the Sun's apparent semi-diameter would be the same at all times of the year; but the Sun's apparent semi-diameter varies in the course of a year, from the greatest $16'.\,17''$ to the least $15'.\,45''$, the maximum and minimum taking place at an interval of half a year, and after a motion of 180^0 of longitude. The semi-diameter of the Sun at the Earth's mean distance $= 16'.\,0'',9$. The Sun's diameter, on account of the greater accuracy of the method, is not observed directly, but is calculated from the observed time of transit of the Sun's disk over the meridian. If t be this time, and T the time between two successive transits, $T : t :: 360^0$: Sun's diameter, supposing the Sun to be in the equator. When the Sun is not in the equator, since the arc it describes is part of a small circle, the diameter found as above must be multiplied by the cosine of declination. We shall, in the next Chapter, investigate accurately the sidereal time occupied by the passage of the Sun's diameter across the meridian.

306. *The curve which the Earth describes round the Sun is an ellipse.*

Let Δ and Δ' be the greatest and least apparent diameters of the Sun, δ any other diameter, $a - ae$, $a + ae$, and ρ the corresponding distances of the Earth and Sun, and $l, l + 180^0$,

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* Strictly speaking, the ecliptic at any time is the plane passing through the Earth's center and the direction of the Sun's motion.

$l+v$ the corresponding longitudes of the Sun. Then, by comparing the observed longitudes and diameters of the Sun at different times, it appears that $\Delta-\delta$, the decrement of the Sun's diameter, varies as the versed-sine of the difference of the Sun's longitudes, or as $1-\cos v$;

$$\therefore \frac{\Delta-\Delta'}{\Delta-\delta}=\frac{1-\cos 180^\circ}{1-\cos v};$$

$$\therefore (\Delta-\Delta').(1-\cos v)=2\Delta-2\delta;$$

$$\therefore \delta=\Delta-\tfrac{1}{2}(\Delta-\Delta').(1-\cos v)$$

$$=\tfrac{1}{2}(\Delta+\Delta')+\tfrac{1}{2}(\Delta-\Delta').\cos v;$$

or, substituting for δ, Δ, Δ', the reciprocals of ρ, $a-ae$, $a+ae$, to which they are proportional,

$$\frac{1}{\rho}=\frac{1}{a(1-e^2)}+\frac{e}{a(1-e^2)}.\cos v.$$

This is the well known equation to an ellipse having the Earth in its focus, and whose major axis passes through the places of the greatest and least apparent diameters.

307. The angle described by the Earth round the Sun from perihelion in any time t, is called the *True Anomaly* (v). The angle described round the Sun in the same time (t) by a body which moves with the Earth's mean angular velocity, i. e. which completes a revolution round the Sun by a uniform angular motion in the same time as the Earth does by its variable one, is called the *Mean Anomaly* (m); therefore, if P be the time of the Earth's period, or one year,

$$m=360^\circ.\frac{t}{P}.$$

The anomalies are often reckoned from aphelion instead of perihelion. Any expressions depending upon the anomalies reckoned from perihelion may be changed into equivalent expressions where those quantities are reckoned from aphelion, by substituting for v and m, $180^\circ-v$ and $180^\circ-m$ respectively; or by simply changing the sign of e, since the fundamental equation

$$\rho = \frac{a(1-e^2)}{1+e\cos v} \text{ equally becomes } \rho = \frac{a(1-e^2)}{1-e\cos v}$$

whether v be changed into $180^0 - v$, or e be made negative.

308. *To find the motion of the aphelion.*

The Earth's motion in longitude is the same at equal distances from the aphelion; if therefore L be any longitude of the Earth, before, and L' the longitude after passing the aphelion when the Earth's motion (or the Sun's apparent motion) is the same as before, $\frac{1}{2}.(L' + L)$ is the longitude of the aphelion, supposing the equinox from which L and L' are measured to remain fixed during the interval. If p be the precession of the equinoxes whilst the Earth's longitude increases from L to L', $\frac{1}{2}.(L' - p + L)$ is the longitude of the aphelion. From comparing the longitude of the aphelion determined in this manner, with its value found after an interval of four or five centuries, it has been found that the longitude of the aphelion has an annual increase of 61″,47, from which, subtracting the precession, there remains 11″,25 nearly for the annual motion of the aphelion.

Although the above method does not ascertain the place of the aphelion very correctly, on account of the difficulty of observing when the Sun's motions are exactly equal, yet, on account of the great interval between the two determinations of the aphelion, the motion of that point is determined with great accuracy. or in 115,200 yrs

309. *To find the longitude of the perihelion, and the time of the Earth's passing through it.*

Let EAP (fig. 46.) be the Earth's orbit round the Sun S, AP the line of apsides, E the Earth *near* the aphelion at the time T, E' the Earth near the perihelion at the time T', Aa, Pp the motions of A and P whilst the Earth moves from E to E'. Now, the Earth's daily motions in longitude at E and E' being near the maximum and minimum, may be considered uniform; if, therefore, they be represented by n and p, and x and y be the respective times of the Earth's moving from E to A, and from E' to p, $ASE = nx$, $E'Sp = py$. Let $ES♈$ the longitude of $E = L$, $E'S$♈$'$ the longitude of $E' = L'$; therefore, since the time from E to E' is half a year nearly, ♈S♈$' = 25''$, and we have

the longitude of the Earth at $A = L + nx$,

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots P = \text{long. at } p - 6''$$

$$= L' + py - 6'';$$

$$\therefore \text{ long. at } P - \text{long. at } A = L' - L + py - nx - 6'',$$

$$\text{or } \; 180^0 + 25'' = L' - L + (p - n) \cdot y + n \cdot (y - x) - 6'';$$

$$\therefore \; y = \frac{180^0 + 31'' + L - L' - n \cdot (y - x)}{p - n}.$$

But $182^d.15^h.6^m.52^s$, which is half the time of an anomalistic year (M), = the time of the Earth's moving from perihelion to aphelion

$$= T' + y - (T + x) = T' - T + (y - x);$$

$$\therefore \; y - x = M - T' + T,$$

$$\text{and } \; y = \frac{180^0 + 31'' + L - L' - n \cdot (M - T' + T)}{p - n};$$

therefore, the longitude of the perihelion at the time T'

$$= L' + py,$$

and the time of the Earth's being in the perihelion

$$= T' + y,$$

are both determined.

310. Hence the longitude of the perigee $= L' + py - 180^0$. Also if π be the longitude of the perihelion, and l the longitude of the Earth seen from the Sun, or, as it is called, the *heliocentric* longitude of the Earth, then the corresponding true anomaly $v = l - \pi$.

311. *To find the eccentricity of the orbit described by the Earth round the Sun.*

Let e be the eccentricity; then (n and p being as in the last Article), $\frac{1}{2} a^2 (1 + e)^2 n$ and $\frac{1}{2} a^2 (1 - e)^2 . p$ are the areas described round the Sun in equal times, when the Earth is in the aphelion and perihelion, and, as these are equal,

$$\therefore\ (1+e)\sqrt{n} = (1-e)\sqrt{p};$$

$$\therefore\ e = \frac{\sqrt{p}-\sqrt{n}}{\sqrt{p}+\sqrt{n}}.$$

Second Method. e may also be determined from the greatest and least apparent semi-diameters of the Sun; if these are Δ and Δ',

$$\frac{1+e}{1-e} = \frac{\Delta}{\Delta'};\quad \therefore\ e = \frac{\Delta-\Delta'}{\Delta+\Delta'}.$$

Hence, taking the values of Δ, Δ', in Art. 305, we get $e = \cdot 0167$ nearly.

312. *To find the mean distance of the Earth from the Sun.*

Let a be the mean distance of the Earth from the Sun, ρ any other distance, v the corresponding true anomaly, measured from perihelion; then, if l be the heliocentric longitude of the Earth deduced from an observed longitude of the Sun, and π the longitude of the perihelion, $v = l - \pi$,

$$\therefore\ \rho = \frac{a(1-e^2)}{1+e\,.\cos(l-\pi)}.$$

In this equation e, l, and π are known; therefore, if any distance ρ of the Sun from the Earth could be found, a would be known. A mode of determining accurately the Sun's parallax will be explained in a subsequent Chapter, and from this ρ, and, therefore, a may be obtained.

313. *In an elliptical orbit of small eccentricity described about the Sun, the angular motion is nearly uniform round the focus in which the Sun is not.*

Let P (fig. 47.) describe an ellipse of small eccentricity round S, and let H be the other focus; let $PHA = \phi$, and let, a, e, v be as before.

$$\text{Then } \frac{a(1-e^2)}{1+e\cos v} + \frac{a(1-e^2)}{1-e\cos\phi} = SP + HP = 2a;$$

$$\therefore \frac{a(1-e^2)\,e\sin v}{(1+e\cos v)^2}\,.\,d_t v - \frac{a(1-e^2)\,e\sin\phi}{(1-e\cos\phi)^2}\,.\,d_t\phi = 0.$$

$$\text{Or } SP^2\,.\,\sin v\,.\,d_t v - HP^2\,.\,\sin\phi\,.\,d_t\phi = 0;$$

$$\therefore SP^2\,.\,d_t v = HP^2\,.\,\frac{\sin\phi}{\sin v}\,.\,d_t\phi = HP^2\,\frac{SP}{HP}\,d_t\phi.$$

But $\frac{1}{2}\,SP^2\,.\,d_t v$ (= limit of the ratio of the area described round S and corresponding time) is constant,

$$\therefore SP\,.\,HP\,.\,d_t\phi \text{ is constant};$$

$$\therefore d_t\phi \propto \frac{1}{SP\,.\,HP} \propto \frac{1}{a^2 - e^2x^2} \propto \frac{1}{a^2} + \frac{x^2e^2}{a^4} + \text{\&c.} \propto \frac{1}{a^2} \text{ nearly,}$$

since ex is very small compared with a. Hence, the motion round H is nearly uniform, and much more so than that round $S(d_t v)$, which varies as $\frac{1}{SP^2}$, or as $\frac{1}{\overline{a+ex}\big|^2}$, that is, as $\frac{1}{a^2} - \frac{2xe}{a^3}$ nearly.

314. *Given the mean, to find the true anomaly, in orbits of small eccentricity.*

The same construction remaining, since the angular motion round H is nearly uniform, take AHP (fig. 47.) for the mean anomaly (m), then ASP is the true anomaly; produce HP to D, making $PD = PS$, or $HD = 2a$, and join SD. Then

$$\tan\tfrac{1}{2}(HSD + HDS) = \frac{HD + HS}{HD - HS}\tan\tfrac{1}{2}(HSD - HDS)$$

$$\text{or } \tan\tfrac{1}{2}(180^\circ - m) = \frac{1+e}{1-e}\tan\tfrac{1}{2}(180^\circ - v),$$

$$\text{or } \tan\tfrac{1}{2}v = \frac{1+e}{1-e}\tan\tfrac{1}{2}m,$$

which determines v from m, very nearly, in orbits of small eccentricity.

315. The relation between the true and the mean anomaly may be deduced simply from geometrical considerations.

Let P (fig. 48.) be the Earth, S the Sun, APB the ellipse described, C its center, PN an ordinate to the axis meeting the circle described on AB in Q; join SQ, SP, CQ; let ASP, ACM the true and mean anomalies be measured from the *perihelion* A, and let $ACQ = u$, which angle is called the *eccentric* anomaly.

Then if $CN = x$, and t be the time from perihelion to P,

$$m = t \cdot \frac{2\pi}{P} = \frac{\text{area } ASP}{\text{area of ellipse}} \cdot 2\pi = \frac{\text{area } ASQ}{\pi a^2} \cdot 2\pi.$$

But area ASQ = area ACQ − triangle SCQ

$$= \tfrac{1}{2} a^2 u - \tfrac{1}{2} ae \cdot a \sin u;$$

$$\therefore m = u - e \sin u \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again $\rho = a + ex = a \cdot (1 - e \cos u)$;

$$\therefore \cos v = -\frac{ae + x}{\rho} = \frac{a \cos u - ae}{a(1 - e \cos u)} = \frac{\cos u - e}{1 - e \cos u};$$

$$\therefore \frac{1 - \cos v}{1 + \cos v} = \frac{1 + e - (1 + e)\cos u}{1 - e + (1 - e)\cos u},$$

$$\text{or } \tan^2 \tfrac{1}{2} v = \frac{1 + e}{1 - e} \cdot \tan^2 \tfrac{1}{2} u;$$

$$\therefore \tan \tfrac{1}{2} v = \sqrt{\frac{1 + e}{1 - e}} \cdot \tan \tfrac{1}{2} u \ldots\ldots\ldots (2).$$

Therefore, eliminating u between the equations (1) and (2), the relation between m and v may be found.

316. To find the true anomaly in terms of the mean, in a series ascending by powers of e.

Let v be the true and m the mean anomaly at any time t (t being expressed in days) from perihelion; then, P being $365^{\text{d}}.2595$, we have

$$d_t m = \frac{2\pi}{P}, \quad \text{and } d_t v = \frac{2\pi\, a b}{P} \cdot \frac{1}{\rho^2};$$

$$\therefore\ d_m v = \frac{a b}{\rho^2}$$

$$= \sqrt{1 - e^2}\, (1 + 2e \cos u + 3e^2 \cos^2 u + 4e^3 \cos^3 u + \&c.),$$

$$\text{since } \rho = a\,(1 - e \cos u).$$

Now by Lagrange's Theorem, if $u = m + e \sin u$,

$$\text{then } f(u) = f(m) + \sin m\, f'(m) \,.\, e + \left\{\sin^2 m\, f'(m)\right\}' \frac{e^2}{2}$$

$$+ \left\{\sin^3 m f'(m)\right\}'' \frac{e^3}{6} + \&c.$$

$$\therefore\ \cos u = \cos m + \sin m\,(-\sin m)\; e + \tfrac{1}{4}\,(\sin 3m - 3 \sin m)' \frac{e^2}{2} + \&c.$$

$$= \cos m - (1 - \cos 2m)\,\frac{e}{2} + (\cos 3m - \cos m)\,\frac{3e^2}{8} + \&c.$$

$$\cos^2 u = \cos^2 m + \sin m\ (-\sin 2m)\ e + \&c.$$

$$= \tfrac{1}{2}\,(1 + \cos 2m) - (\cos m - \cos 3m)\,\frac{e}{2} + \&c.$$

$$\cos^3 u = \cos^3 m = \tfrac{1}{4}\,(\cos 3m + 3 \cos m) + \&c.;$$

$$\therefore\ d_m v = (1 - \tfrac{1}{2}e^2)\,\Big\{1 + 2e \cos m - (1 - \cos 2m)\, e^2$$

$$+ (\cos 3m - \cos m)\,\frac{3e^3}{4} + (1 + \cos 2m)\,\frac{3e^2}{2}$$

$$+ (\cos 3m - \cos m)\,\frac{3e^3}{2} + (\cos 3m + 3 \cos m)\, e^3 + \&c.\Big\}$$

$$= 1 + 2e \cos m + \frac{5e^2}{2} \cos 2m + \frac{13}{4} e^3 \cos 3m - \frac{e^3}{4} \cos m + \&c.;$$

$$\therefore\ v = m + \left(2e - \frac{e^3}{4}\right) \sin m + \frac{5e^2}{4} \sin 2m + \frac{13}{12} e^3 \sin 3m + \&c.$$

317. *To express the radius vector in terms of the mean anomaly in a series ascending by powers of e.*

$$\rho = a\,(1 - e\cos u) = f(u), \quad u = m + e\sin u\,;$$

$$\therefore\ \rho = f(m) + \{\sin m \,.\, f'(m)\}\,e + \{\sin^2 m \,.\, f'(m)\}'\,\frac{e^2}{2}$$

$$+ \{\sin^3 m \,.\, f'(m)\}''\,\frac{e^3}{6} + \&c.$$

$$\text{but } f(m) = a\,(1 - e\cos m), \quad f'(m) = ae\sin m,$$

$$\sin m \,.\, f'(m) = ae\sin^2 m = \tfrac{1}{2}\,ae\,(1 - \cos 2m),$$

$$\{\sin^2 m \,.\, f'(m)\}' = (ae\sin^3 m)' = 3ae\sin^2 m\cos m = \frac{3ae}{2}\sin m\sin 2m$$

$$= \frac{3ae}{2^2}\,(\cos m - \cos 3m),\ \&c.\,;$$

$$\therefore\ \rho = a\,(1 - e\cos m) + \tfrac{1}{2}\,ae\,(1 - \cos 2m)\,e$$

$$+ \frac{3ae}{4}\,(\cos m - \cos 3m)\,\frac{e^2}{2} + \&c.$$

$$= a\,\left\{1 + \tfrac{1}{2}e^2 - e\cos m - \frac{e^2}{2}\cos 2m - \frac{e^3}{2}\,.\,\frac{3}{4}\,(\cos 3m - \cos m) + \&c.\right\}.$$

318. *To expand the mean anomaly in terms of the true.*

$$d_v m = \frac{\rho^2}{ab} = \frac{(1 - e^2)^{\frac{3}{2}}}{(1 + e\cos v)^2} = (1 - e^2)^{\frac{3}{2}}\,d_e\left(\frac{e}{1 + e\cos v}\right);$$

let $1 + e\cos v = \alpha^2 + 2\alpha\beta\cos v + \beta^2 = \alpha^2\,(1 + cx)\,(1 + cx^{-1})$,

$$\text{making } c = \frac{\beta}{\alpha}, \quad 2\cos v = x + x^{-1}\,;$$

therefore, since $\alpha^2 + \beta^2 = 1$, $2\alpha\beta = e$,

$$c = \frac{\beta}{\alpha} = \frac{\sqrt{1 + e} - \sqrt{1 - e}}{\sqrt{1 + e} + \sqrt{1 - e}} = \frac{e}{1 + \sqrt{1 - e^2}}\,;$$

$$\therefore \quad \frac{1}{1 + e\cos v} = \frac{1}{a^2} \frac{1}{(1 + cx)\,(1 + cx^{-1})}$$

$$= \frac{1}{a^2\,(1 - c^2)} \left(\frac{1}{1 + cx} - \frac{cx^{-1}}{1 + cx^{-1}}\right)$$

$$= \frac{1}{a^2 - \beta^2}\,(1 - cx + c^2x^2 - \&c. - cx^{-1} + c^2x^{-2} - c^3x^{-3} + \&c.)$$

$$= \frac{1}{\sqrt{1 - e^2}}\,(1 - 2c\cos v + 2c^2\cos 2v - 2c^3\cos 3v + \&c.)$$

Hence the general term of the expansion of $d_e\left(\frac{e}{1 + e\cos v}\right)$ is

$$\pm\, 2\cos rv\, d_e\left(\frac{ec^r}{\sqrt{1 - e^2}}\right) = \pm\, 2\cos rv\,\frac{c^r\,(1 + r\sqrt{1 - e^2})}{(1 - e^2)^{\frac{3}{2}}}\,;$$

$$\therefore\ d_v m = 1 - 2c\,(1 + \sqrt{1 - e^2})\cos v + 2c^2\,(1 + 2\sqrt{1 - e^2})\cos 2v$$

$$- \&c. + 2\,(-c)^r\,(1 + r\sqrt{1 - e^2})\cos rv + \&c.\,;$$

$$\therefore\ m = v - 2c\,(1 + \sqrt{1 - e^2})\sin v + c^2\,(1 + 2\sqrt{1 - e^2})\sin 2v.$$

$$- \&c. + 2\,(-c)^r\,(1 + r\sqrt{1 - e^2})\,\frac{1}{r}\sin rv + \&c.$$

The equation of the center, which is the difference between the true and mean anomalies, may be obtained from Art. 316, in terms of m, and from the present Art. in terms of v.

319. In orbits of small eccentricity, *AHP*, fig. 47, represents the mean anomaly, therefore *SPH* will represent $v - m$ or the equation of the center;

$$\text{but}\ \sin(v - m) = \frac{SH}{SP}\sin m = \frac{2e\sin m}{1 + ex},$$

$$\therefore\ v - m = 2e\sin m\ \text{nearly},$$

the maximum value of which is $2e$; therefore in orbits of small eccentricity, *the greatest equation of the center has twice the eccentricity for its circular measure.*

If the anomalies are measured from the apogee, the greatest equation of the center is the greatest value of $m - v$, which $=$ $-$ the greatest value of $v - m$; therefore the greatest equation of the center must change its sign, when the anomalies are measured from the apogee; that is, when e is changed into $-e$; hence the greatest equation of the center can contain only the odd powers of e, and the preceding result which is obtained by neglecting e^2, is the same as would have been found by neglecting e^3, and higher powers.

320. *Given the time of the year, to find the Sun's longitude, and conversely, given the longitude, to find the time.*

Let T be the time, t the time of the Sun's passing the perigee, s the Sun's mean daily motion, then $s.(T-t)$ is the Sun's mean anomaly, and if to this be added the equation of the center E, we get $s(T-t)+E$ for the true anomaly; and therefore $s.(T-t)+E+p$ for the Sun's true longitude l, p being the longitude of the perigee;

$$\therefore \quad T = t + \frac{l-p-E}{s}.$$

321. As the planets revolve round the Sun in the same manner as the Earth, in orbits of small eccentricity, the preceding equations may be applied to the determination of the planetary motions.

If m be the mean anomaly in a planet's orbit corresponding to a time t from perihelion, and P' the periodic time, we have, as for the Earth,

$$m = \frac{360^0}{P'}.t.$$

But if a' be the mean distance of the planet, $P' = \left(\frac{a'}{a}\right)^{\frac{3}{2}}.1^{\text{y}}$,

$$\therefore \quad m = 360^0.\left(\frac{a}{a'}\right)^{\frac{3}{2}}.t,$$

t being expressed in parts of a year, and a being the Earth's mean distance.

The orbits described by comets are very eccentric ellipses, which, with few exceptions, may be considered as parabolas for those parts during the description of which the comet is visible from the Earth. The consideration of the circumstances of a body's motion in a very eccentric ellipse, is not sufficiently elementary for the present work. The following propositions will, however, be of use in the next chapter.

322. *To find the radius vector and the time, corresponding to a given true anomaly, in an orbit where* e = 1, *nearly.*

$$\rho = \frac{a(1-e^2)}{1+e\cos v} = \frac{a(1-e^2)}{1+e\left(\cos^2\frac{v}{2}-\sin^2\frac{v}{2}\right)}$$

$$= \frac{a(1-e^2)}{\cos^2\frac{v}{2}\left\{1+e+(1-e)\tan^2\frac{v}{2}\right\}} = \frac{a(1-e^2)}{\cos^2\frac{v}{2}\left\{1+e+(1+e)\tan^2\frac{u}{2}\right\}},$$

$$\text{or } \rho = \frac{a(1-e)\cos^2\frac{1}{2}u}{\cos^2\frac{1}{2}v} = p\,.\,\frac{\cos^2\frac{1}{2}u}{\cos^2\frac{1}{2}v}\ldots\ldots\ldots(1),$$

p being the perihelion distance $a - ae$.

Let $c = 1 - e$, where, since e is nearly $= 1$, c is very small; therefore $p = ac$; also suppose t expressed in years, and a in parts of the Earth's mean distance from the Sun;

$$\therefore\ d_v t = d_m t\,.\,d_v m = \frac{a^{\frac{3}{2}}}{2\pi}\cdot\frac{(1-e^2)^{\frac{3}{2}}}{(1+e\cos v)^2} = \frac{p^{\frac{3}{2}}}{2\pi c^{\frac{3}{2}}}\cdot\frac{(2c-c^2)^{\frac{3}{2}}}{(1+e\cos v)^2},$$

$$\text{let } x = \tan\frac{v}{2};\quad \therefore\ d_x v = \frac{2}{1+x^2},$$

$$\text{and } 1+e\cos v = 1+(1-c)\frac{1-x^2}{1+x^2} = \frac{2-c(1-x^2)}{1+x^2};$$

$$\therefore\ d_x t = \frac{p^{\frac{3}{2}}(2-c)^{\frac{3}{2}}(1+x^2)}{\pi\left\{2-c(1-x^2)\right\}^2}$$

$$= \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}}\left(1-\frac{3c}{4}\right)(1+x^2)\left\{1+c(1-x^2)+\frac{3c^2}{4}(1-x^2)^2+\&c.\right\}$$

$$= \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}}\left(1-\frac{3c}{4}\right)\cdot\left\{1+x^2+c(1-x^4)\right\};$$

$$\therefore\ t = \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}}\left(1 - \frac{3c}{4}\right).\left\{x + \frac{x^3}{3} + c\left(x - \frac{x^5}{5}\right)\right\}$$

$$= \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}}\left\{x + \frac{x^3}{3} + c\left(\frac{x}{4} - \frac{x^3}{4} - \frac{x^5}{5}\right)\right\}$$

$$= \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}}\left\{\tan\frac{v}{2} + \frac{1}{3}\tan^3\frac{v}{2} + \frac{c}{4}\left(\tan\frac{v}{2} - \tan^3\frac{v}{2} - \frac{4}{5}\tan^5\frac{v}{2}\right)\right\}\ldots (2).$$

323. When c is 0, the curve described becomes a parabola, and the equations (1) and (2) become

$$\rho = \frac{p}{\cos^2\frac{1}{2}v},$$

$$t = \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}}.\left\{\tan\tfrac{1}{2}v + \tfrac{1}{3}\tan^3\tfrac{1}{2}v\right\},$$

which may be applied to the determination of the motions of comets, since very few of their paths differ sensibly from parabolas, whilst they are near the Earth's orbit.

324. *To find the time of describing any arc of a parabola in terms of its chord and the focal distances of its extremities.*

Let ρ, ρ', be the two focal distances, c the chord joining their extremities, v, v', the corresponding true anomalies, a the focal distance of the vertex;

$$\therefore\ \rho = a\sec^2\frac{v}{2} = a\,(1 + t^2),\quad \rho' = a\sec^2\frac{v'}{2} = a\,(1 + t'^2),$$

denoting $\tan\frac{v}{2}$, $\tan\frac{v'}{2}$, by t, t', respectively;

then included sectorial area $= \int_v \frac{1}{2}\rho^2 = \int_v \frac{a^2}{2}\sec^4\frac{v}{2}$

$$= a^2\int_v\left(1 + \tan^2\frac{v}{2}\right)d_v\tan\frac{v}{2},\ \text{(between the limits } v = v',\ v = v,)$$

$$= a^2\left\{t' - t + \frac{1}{3}\left(t'^3 - t^3\right)\right\} = \frac{a^2}{3}\left(t' - t\right)\left(1 + tt' + 1 + t^2 + 1 + t'^2\right).$$

Now $v' - v$ being the angle of a triangle contained by the sides ρ', ρ, and subtended by the side c,

$$\cos \tfrac{1}{2}(v' - v) = \sqrt{\frac{s(s-c)}{\rho\rho'}}, \text{ where } 2s = \rho + \rho' + c;$$

$$\therefore\ 1 + tt' = \cos \tfrac{1}{2}(v' - v) \,.\, \sec \tfrac{1}{2} v \,.\, \sec \tfrac{1}{2} v'$$

$$= \sqrt{\frac{s(s-c)}{\rho\rho'}} \cdot \frac{\sqrt{\rho\rho'}}{a} = \frac{1}{a}\sqrt{s(s-c)},$$

$$\text{and } (t' - t)^2 = 1 + t^2 + 1 + t'^2 - 2(1 + tt') = \frac{1}{a}\left\{\rho + \rho' - 2\sqrt{s(s-c)}\right\}$$

$$= \frac{1}{a}\left\{s - 2\sqrt{s(s-c)} + s - c\right\};$$

$$\therefore\ t' - t = \frac{1}{\sqrt{a}}(\sqrt{s} - \sqrt{s-c});$$

$$\therefore \text{ area} = \frac{\sqrt{a}}{3}(\sqrt{s} - \sqrt{s-c})\left\{\sqrt{s(s-c)} + \rho + \rho'\right\}$$

$$= \frac{\sqrt{a}}{3}(\sqrt{s} - \sqrt{s-c})\left\{s + \sqrt{s(s-c)} + s - c\right\}$$

$$= \frac{\sqrt{a}}{3}\left\{s^{\frac{3}{2}} - (s-c)^{\frac{3}{2}}\right\} = \frac{\sqrt{a}}{6\sqrt{2}}\left\{(\rho + \rho' + c)^{\frac{3}{2}} - (\rho + \rho' - c)^{\frac{3}{2}}\right\}.$$

But the area described in the unit of time about the focus $= \sqrt{\dfrac{ma}{2}}$, where m is the force at the unit of distance; therefore the time of describing the arc

$$= \frac{1}{6\sqrt{m}}\left\{(\rho + \rho' + c)^{\frac{3}{2}} - (\rho + \rho' - c)^{\frac{3}{2}}\right\}.$$

325. If the extremities of the arc fall on different sides of the vertex the expression for the time is

$$\frac{1}{6\sqrt{m}}\left\{(\rho + \rho' + c)^{\frac{3}{2}} + (\rho + \rho' - c)^{\frac{3}{2}}\right\}.$$

If a day be taken for the unit of time, and the radius of the Earth's orbit for the unit of distance,

$$365\tfrac{1}{4} = \frac{\text{area of } \oplus\text{'s orbit}}{\tfrac{1}{2} \cdot 1 \cdot \text{vel. of } \oplus} = \frac{\pi}{\tfrac{1}{2}\sqrt{m}}; \quad \therefore \frac{1}{6\sqrt{m}} = \frac{365\tfrac{1}{4}}{12\pi},$$

and the time of describing the arc

$$= \frac{365\tfrac{1}{4}}{12\pi}\left\{(\rho + \rho' + c)^{\frac{3}{2}} - (\rho + \rho' - c)^{\frac{3}{2}}\right\}.$$

326. *To find the time of describing any arc of an ellipse in terms depending on its chord, and the focal distances of its extremities, and the major axis of the orbit.*

Let ρ, ρ', be the two focal distances, c the chord joining their extremities, x, x', y, y', the co-ordinates of the extremities measured from the focus; v, v', u, u', the corresponding true and eccentric anomalies; a, b, the semi-axes of the orbit. Then included sectorial area $= \frac{1}{2} ab \left\{u' - u - e(\sin u' - \sin u)\right\}$.

Let $\rho + \rho' + c = 2a(1 - \cos\phi')$, $\rho + \rho' - c = 2a(1 - \cos\phi)$;

therefore, $v' - v$ being the angle of a triangle contained by the sides ρ', ρ, and subtended by c,

$$\rho\rho' \cos^2\tfrac{1}{2}(v' - v) = \tfrac{1}{2}(\rho + \rho' + c) \cdot \tfrac{1}{2}(\rho + \rho' - c) = a^2(1 - \cos\phi')(1 - \cos\phi);$$

$$\text{but } \rho\rho' = a^2\left\{1 - e(\cos u + \cos u') + e^2 \cos u \cos u'\right\}$$

$$\text{and } \rho\rho' \cos(v' - v) = \rho\rho' \cos v \cos v' + \rho\rho' \sin v \sin v' = xx' + yy'$$

$$= a^2(e - \cos u)(e - \cos u') + a^2(1 - e^2) \sin u \sin u'$$

$$= a^2\left\{\cos(u' - u) - e(\cos u + \cos u') + e^2(1 - \sin u \sin u')\right\};$$

$$\therefore \rho\rho'\left\{1 + \cos(v' - v)\right\}$$

$$= a^2\left\{1 + \cos(u' - u) - 2e(\cos u + \cos u') + e^2[1 + \cos(u + u')]\right\},$$

$$\text{or } \rho\rho' \cos^2\tfrac{1}{2}(v' - v)$$

$$= a^2\left\{\cos^2\tfrac{1}{2}(u' - u) - 2e\cos\tfrac{1}{2}(u' - u)\cos\tfrac{1}{2}(u' + u) + e^2\cos^2\tfrac{1}{2}(u' + u)\right\};$$

$$\therefore \left\{\cos\tfrac{1}{2}(u' - u) - e\cos\tfrac{1}{2}(u' + u)\right\}^2$$

$$= (1 - \cos\phi')(1 - \cos\phi) = (2\sin\tfrac{1}{2}\phi \sin\tfrac{1}{2}\phi')^2 \ldots\ldots (1).$$

$$\text{Also } \frac{1}{a}(\rho + \rho') = 2 - \cos\phi' - \cos\phi = 2 - e(\cos u + \cos u'),$$

$$\therefore \cos\phi' + \cos\phi = e\,(\cos u' + \cos u)$$

$$\text{or } 4e\cos\tfrac{1}{2}(u'+u)\cos\tfrac{1}{2}(u'-u) = 2\,(\cos\phi' + \cos\phi);$$

adding this to equation (1)

$$\left\{\cos\tfrac{1}{2}(u'-u) + e\cos\tfrac{1}{2}(u'+u)\right\}^2$$

$$= (1+\cos\phi')(1+\cos\phi) = (2\cos\tfrac{1}{2}\phi'\cos\tfrac{1}{2}\phi)^2 \ldots\ldots (2).$$

Hence extracting the roots of equations (1) and (2) and adding and subtracting, we obtain

$$\cos\tfrac{1}{2}(u'-u) = \cos\tfrac{1}{2}(\phi'-\phi),\quad e\cos\tfrac{1}{2}(u'+u) = \cos\tfrac{1}{2}(\phi'+\phi);$$

$$\therefore u' - u = \phi' - \phi, \text{ and } \sin\tfrac{1}{2}(u'-u) = \sin\tfrac{1}{2}(\phi'-\phi);$$

$$\therefore e\sin\tfrac{1}{2}(u'-u)\cos\tfrac{1}{2}(u'+u) = \sin\tfrac{1}{2}(\phi'-\phi)\cos\tfrac{1}{2}(\phi'+\phi),$$

$$\text{or } e\,(\sin u' - \sin u) = \sin\phi' - \sin\phi;$$

$$\therefore \text{area} = \tfrac{1}{2}ab\,\{\phi' - \phi - (\sin\phi' - \sin\phi)\}.$$

But the sectorial area described in the unit of time $= \dfrac{\pi ab}{P}$; therefore the time of describing the sector

$$= \frac{P}{2\pi}\{\phi' - \phi - (\sin\phi' - \sin\phi)\},$$

ϕ and ϕ' depending upon the focal distances and the chord by the equations

$$\sin\tfrac{1}{2}\phi' = \tfrac{1}{2}\left(\frac{\rho'+\rho+c}{a}\right)^{\frac{1}{2}},\quad \sin\tfrac{1}{2}\phi = \tfrac{1}{2}\left(\frac{\rho'+\rho-c}{a}\right)^{\frac{1}{2}}.$$

CHAPTER XIII.

THE EQUATION OF TIME.

327. To find the equation of time.

Let Z (fig. 13, Plate I.) be the zenith of the place of observation, P the pole of the equator, K that of the ecliptic,

Ω = longitude of the Moon's node,

Υ the mean first point of Aries,

Υ' the true first point of Aries,

S the Sun, M the mean place of the Sun,

$\therefore$ ΥM = mean Sun's mean longitude.

T a point the right ascension of which, measured from the mean first point of Aries, equals the mean Sun's mean longitude, $\therefore TP\Upsilon = M\Upsilon$, E = equation of the center, P = planetary perturbations, $R = S\Upsilon' - SP\Upsilon'$ = reduction to equator at S.

Then (Art. 280),

$$SP\Upsilon' - SP\Upsilon = -16'',8 \sin \Omega \cos \omega,$$

$$S\Upsilon' - S\Upsilon = -16'',8 \sin \Omega$$

$$S\Upsilon = M\Upsilon + E + P.$$

Now 15 (equation of time) = 15 (mean solar time − true solar time)

$$= TPZ - SPZ = SP\Upsilon - TP\Upsilon;$$

$$\text{but } SP\Upsilon = SP\Upsilon' + 16'',8 \sin \Omega \cos \omega$$

$$= S\Upsilon' - R + 16'',8 \sin \Omega \cos \omega$$

$$= S\Upsilon - R - 16'',8 \sin \Omega\ (1 - \cos \omega)$$

$$\text{and } TP\Upsilon = M\Upsilon;$$

$$\therefore 15 \text{ (equation of time)} = E + P - R - 16'',8 \sin \Omega\ (1 - \cos \omega).$$

Hence, more strictly, the equation of time is the difference of the Sun's true right ascension and mean longitude, corrected by the equation of the equinoxes in right ascension.

The true longitude may be obtained in terms of the mean, or conversely, by the equations of the preceding Chapter, and thus the equation of time may be determined in terms of the true or mean longitude of the Sun; Littrow, (Vol. II. p. 71.) has given expressions in which the equation of time is expressed in a series of sines and cosines of multiple arcs of the true, and also of the mean longitude of the Sun, from which tables may be constructed*.

328. The equation of time in the Nautical Almanac is given only for the time when the Sun is on the meridian of Greenwich, and for any other time, or when the Sun is on any other meridian, the proportional part must be taken. Let E and E' be the values of the equation of time on two successive noons, taken from the Nautical Almanac, $E + \Delta E$ the equation of time at some intermediate instant, when the Sun is on a meridian h^0 west of Greenwich. Then, the increment of E may be considered proportional to the time, or to the hour angle described by the Sun nearly;

$$\therefore \Delta E = \frac{h^0}{360^0} \cdot (E' - E);$$

$$\therefore E + \Delta E = E + \frac{h^0}{360^0} \cdot (E' - E).$$

329. From the expression for the equation of time (Art. 327), it appears that if the Sun moved uniformly in the ecliptic,

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* If E be the equation of time, calculated upon the supposition that the eccentricity of the Earth's orbit $= .016791$, the longitude of the perihelion $(p) = 279^0.29'.33''$, and the obliquity $= 23^0.27'.57''$, which values are nearly sufficient from the year 1800 to 1900,

$$\begin{aligned} E = 461''.786 \sin (l-p) &- 593.146 \sin 2l \\ -2.907 \sin 2(l-p) &+ 12.793 \sin 4l \\ +0.022 \sin 3(l-p) &- 0.368 \sin 6l \\ +0.099 \sin \Omega &+ \frac{1}{15} P. \end{aligned}$$

$$\begin{aligned} \text{Or } E = 79''.36 \sin L &- 597.08 \sin 2L - 3.42 \sin 3L \\ + 13.25 \sin 4L &+ 0.15 \sin 5L - 0.40 \sin 6L \\ + 435.8 \cos L &+ 1.6 \cos 2L - 18.8 \cos 3L \\ - 0.2 \cos 4L &+ 0.9 \cos 5L \\ + 0.1 \sin \Omega &- 0.1 \sin (2L + \Omega) + \frac{1}{15} P. \end{aligned}$$

or $E = 0$, the equation of time would still vary on account of the variable reduction R, and if the ecliptic coincided with the equator, in which case R would equal 0, the equation of time would change from changes in the quantity E; therefore, the equation of time may be considered as arising from two causes; the obliquity of the ecliptic, and the Sun's unequal motion in longitude. The effects of these may be considered separately, by conceiving a star to move uniformly in the ecliptic with the Sun's mean motion in longitude; then, the difference of the times of transit of this star and the Sun over the meridian, would be the equation of time caused by the Sun's unequal motion in the ecliptic, and the difference of the times of transit of the star, and the fictitious Sun in the equator, would be the equation of time caused by the obliquity.

330. *To compare the Sun's motions in longitude and right ascension.*

Let ω, l, a, δ be, as before, the obliquity, longitude, right ascension and declination of the Sun, at any time t from the vernal equinox.

$$\text{Then } \cos\omega = \tan a \, . \cot l,$$

$$\text{or } \tan a = \cos\omega \, . \tan l;$$

$$\therefore \frac{d_t a}{\cos^2 a} = \frac{\cos\omega \, . \, d_t l}{\cos^2 l} = \frac{\cos\omega \, . \, d_t l}{\cos^2 a \, . \cos^2\delta};$$

$$\therefore d_t a = d_t l \, . \cos\omega \, . \sec^2\delta.$$

If δa, δl be small finite corresponding increments of the Sun's right ascension and longitude,

$$\delta a = \delta l \cos\omega \, . \sec^2\delta, \text{ nearly.}$$

331. *To find when the equation of time caused by the obliquity is additive or subtractive.*

The equation of time is *additive* when it is to be added to apparent time to get the mean time, that is, when mean time precedes apparent time; and *subtractive* in the contrary case.

Let ♈ S (fig. 14, Plate II.) be the ecliptic, ♈ D the equator, SD a circle of declination passing through the Sun S, here supposed to move uniformly in the ecliptic; take $\Upsilon M = \Upsilon S$, then, DM converted into time is the equation of time caused by the obliquity.

Now $\cos \Upsilon SD = \sin \Upsilon \, . \cos \Upsilon D$, which is positive or negative, according as ΥD is less or greater than 90^0; therefore ΥSD is an acute angle when ΥD is less, and obtuse when ΥD is greater than 90^0; but ΥDS is always $= 90^0$; therefore, since the greater side subtends the greater angle, ΥS or ΥM is greater than ΥD when ΥD is less than 90^0, or from an equinox to a solstice, and ΥM is less than ΥD when ΥD is greater than 90^0, or from a solstice to an equinox. At the solstice ΥS and ΥD are each equal to 90^0, and M coincides with D. M also coincides with D at the next equinox, since then ΥS and ΥD are each equal to 180^0. The same effects must take place in the other half of the orbit; therefore, as the Earth revolves round its axis, since its motion of rotation is in the same direction as the Sun's progressive motion through the heavens, a given meridian will pass through M before D or S, from a solstice to an equinox, or mean noon will precede apparent; and through M after D or S from an equinox to a solstice, or mean noon will follow apparent noon; hence the equation of time caused by the obliquity is additive from a solstice to an equinox, evanescent at the solstices and equinoxes, and subtractive from an equinox to a solstice.

332. Since the equation of time, caused by the obliquity is 0 at the equinox and solstice, it must be a maximum at some intermediate point. At this point

$$d_t(l - a) = 0, \text{ or } d_t l = d_t a;$$

$$\therefore 1 = \sec^2 \delta \, . \cos \omega; \text{ (Art. 330)}$$

$$\therefore \cos \delta = \sqrt{\cos \omega}.$$

From this equation δ may be found, and thence l and a, by the equations

$$\sin \delta = \sin \omega \, . \sin l,$$

$$\sin a = \tan \delta \, . \cot \omega.$$

The numerical values of these quantities are

$$l = 46^0.\,14'.\,0'',$$

$$a = 43^0.\,43'.\,50'';$$

therefore, $l - a$, or the maximum equation of time, as far as it arises from this cause, expressed in degrees,

$$= 2^0.\,30',\,10''.$$

333. *To find when the equation of time, caused by the unequal motion in the ecliptic, is additive or subtractive.*

When the Sun is in the perigee, the mean anomaly coincides with the true; but as the Sun is then moving fastest, the motion is greater than the mean motion, and the true place of the Sun precedes the mean till the apogee, when the mean and true places of the Sun coincide. Hence a given meridian, revolving in the same direction, passes through the mean Sun before the true, or mean noon precedes apparent noon. After passing the apogee, the Sun's motion being slower than the mean, the mean place precedes the true, and therefore, mean noon follows apparent till perigee, when they again coincide. Therefore, the equation of time, caused by the Sun's unequal motion in the ecliptic, is additive from perigee to apogee, subtractive from apogee to perigee, and 0 at those points.

334. The greatest separation of the Sun's mean and true places in the ecliptic, or the greatest equation of the center, is $1^0.55'.27'',6$; which will produce the greatest separation of their places referred to the equator, when this quantity, substituted in the equation of Art. 330, for δl, produces the greatest value of δa. In this case,

$$\delta a = (1^0.\,55'.\,27'',6)\cos\omega\,.\,\sec^2\delta,$$

which is greatest when $\delta = \omega = 23^0.\,28'$; therefore, making $\delta = 23^0.\,28'$,

$$\delta a = 2^0.\,5'.\,55'',$$

which, expressed in degrees, is the greatest equation of time that could be caused by the Sun's unequal motion in the ecliptic.

335. *The equation of time vanishes four times a year.*

Let A♈S (fig. 51.) be the ellipse described by the Sun round the Earth, p and a the perigee and apogee, A and ♈ the autumnal and vernal equinoxes, S and W the summer and winter solstices, and M the place (about 46° from ♈), where the equation of time, caused by the obliquity, is a maximum.

Let E be the whole equation of time, t and t' the separate parts caused by the obliquity, and the Sun's elliptical motion respectively; then the maximum value of t is about $2^0. 30'$, (Art. 332), and greater than that of t'. Now, the equation being additive when mean time precedes apparent, by Articles 331, 333,

t is $-$ through AW, ♈S; $+$ through W♈, SaA; and 0 at $\begin{cases} A \\ W \\ \text{♈} \\ S \end{cases}$

t' ....aWp; ....p♈a; 0 at $\begin{cases} p \\ a \end{cases}$

$\therefore$ from A to W, $E = -t - t'$, and at W, $E = -t'$,

from W to p, $E = t - t'$, ... p, $E = t$;

therefore, since E was negative at W and positive at p, there is some point between W and p, at which $E = 0$.

Again,

from p to ♈, $E = +t + t'$, and at ♈, $E = +t'$,

from ♈ to S, $E = -t + t'$, and at S, $E = +t'$.

But since at M, t has its greatest value, which is greater than t'; therefore, at M, E is negative, and it is positive at ♈ and S; therefore, at one point between ♈ and M, and another between M and S, $E = 0$. Lastly

from S to a, $E = +t + t'$, and at a, $E = +t$,

from a to A, $E = +t - t'$, and at A, $E = -t'$,

therefore, at some point between a and A, E must vanish for the fourth time.

The equation of time varies continually in different years, on account of the change in the position of the line of apsides *ap*, which makes at present an angle of about 9^0 with the line of solstices *SW*. If these lines coincided, the equation of time would be nearly 0 at the solstices, since the separate parts of the equation t and t' would each = 0.

336. To find the sidereal time occupied by the passage of the Sun's semi-diameter across the meridian.

Let r be the distance of the Sun from the Earth, the mean distance being 1, and δ his declination at any epoch, ω the obliquity of the ecliptic, and T the length of a tropical year in days; then increase of Sun's longitude in 1 sid. day $= \dfrac{360^0}{r^2(T+1)}$, and therefore (Art. 330) increase of his R. A. $= \dfrac{360^0}{r^2(T+1)} \dfrac{\cos\omega}{\cos^2\delta}$. Hence in one sid. second the Sun's center recedes from ♈ by $\dfrac{15''}{r^2(T+1)} \dfrac{\cos\omega}{\cos^2\delta}$; and in the same time the meridian approaches ♈ by $15''$; consequently in 1 sid. second the meridian separates from the circle of declination passing through the Sun's center by

$$15'' \left(1 - \frac{\cos\omega}{r^2(1+T)\cos^2\delta}\right).$$

Now Sun's semi-diam. at mean dist. = $960'',9$;

$$\therefore \text{ semi-diam. at dist. } r = \frac{960'',9}{r},$$

and angle which this subtends at pole of equator $= \dfrac{960'',9}{r\cos\delta}$.

Let t be the number of sid. seconds which the Sun's semi-diameter takes to pass the meridian;

$$\therefore \frac{960'',9}{r\cos\delta} = 15'' \left(1 - \frac{\cos\omega}{r^2(T+1)\cos^2\delta}\right) t,$$

hence t is known.

CHAPTER XIV.

ON THE PLANETS.

337. The Planets, as seen from the Earth, move among the fixed stars sometimes from west to east, and at other times are stationary, or move from east to west. These appearances, which prove that they do not revolve round the Earth, can all be explained upon the hypothesis of their describing nearly circular orbits round the Sun, subject to the same laws of motion as the Earth.

There are ten planets now known, which are denoted by their appropriate symbols, Mercury (☿), Venus (♀), Mars (♂), Vesta (⚶), Juno (⚵), Ceres (⚳), Pallas (⚴), Jupiter (♃), Saturn (♄), Uranus or the Georgian (♅): of these Mercury, Venus, Mars, Jupiter and Saturn are most conspicuous, and have been observed from the most ancient times; the others, of which Uranus is the chief, are amongst the greatest recent astronomical discoveries. The Earth, considered as a planet, is denoted by the symbol ⊕.

338. The places of the planets as seen from the Earth are called their *Geocentric* places, and are determined by their geocentric latitudes and longitudes computed from the observed right ascensions and declinations. The places as seen from the Sun are called *Heliocentric;* the *Heliocentric Latitude* of a planet being its angular distance from the ecliptic as seen from the Sun, and the *heliocentric longitude* the angle subtended at the Sun between the first point of Aries, and the planet's place referred to the ecliptic.

The angle which the distance of any planet from the Sun subtends at the Earth is called the *Elongation* of the planet.

The intersection of the plane of a planet's orbit with the ecliptic is called the *Line of Nodes;* that point through which the planet passes in moving from the south to the north side of the ecliptic being called the *Ascending Node* (☊), and the other the *Descending Node.* (☋).

The angle at the Sun's center between the radius vector which passes through the ascending node, and the perihelion distance of the planet, is the *Distance of the Perihelion from the Node:* this added to the longitude of the ascending node, is called the *Longitude of the Perihelion.* The angle at the Sun's center between the radius vector of the planet in its orbit, and the radius vector passing through the ascending node, is called the *Argument of Latitude.* The argument of Latitude added to the longitude of the ascending node, is called the *Longitude of the Planet in its Orbit.*

339. If P (fig. 52.) be a planet in its orbit $P\pi ☊$, S the Sun, π the perihelion, $S ☊$ the line of nodes, and $S ♈$ a line passing through the first point of Aries in the plane of the ecliptic ySx; then

$♈ S ☊ = n$, is the longitude of the ascending node,

$\pi S ☊ = \pi$, is the distance of the perihelion from the node,

$PS ☊ = u$, is the argument of latitude;

therefore, if p be the longitude of the perihelion, v the true anomaly, l the longitude of the planet in its orbit, l' the reduced longitude on the ecliptic, and r the reduction, we have from the figure, according to the above definitions,

$$p = n + \pi,$$

$$u = \pi + v = p - n + v,$$

$$l = n + u = p + v,$$

$$l' = l + r.$$

Before proceeding farther, it is necessary to shew how to determine the geocentric place of a planet from the heliocentric, in order to make use of the Tables of the planet's motions, since all the computed motions of a planet are calculated with reference to the Sun: and how to find the heliocentric place from the geocentric, in order to make use of observations which are necessarily made at the Earth.

340. *Having given the heliocentric, to find the geocentric place of a planet at any time.*

Here the heliocentric latitude and longitude, and the heliocentric distance of the planet, and also the position of the Earth, are given, to find the geocentric latitude and longitude and the geocentric distance.

Let S (fig. 53.) be the Sun, and E the Earth in the plane of the ecliptic ySx, P a planet, PN perpendicular to the plane ySx, and NM perpendicular to Sx, $S\Upsilon$, $E\Upsilon$ parallel to each other, passing through the first point of Aries. Let λ ($NE\Upsilon$), β (NEP), ρ (EP) be the geocentric longitude, latitude, and distance of P, respectively; l ($NS\Upsilon$), b (NSP), r (SP), the corresponding heliocentric longitude, latitude and distance of P; L ($ES\Upsilon$), R (SE) the heliocentric longitude of the Earth, and the distance of the Earth from the Sun. Let also r' and ρ', which are called the *curtate distances*, be the projections of r and ρ on the ecliptic, i. e. SN and EN. Then we have

$$r' = r\cos b,\ \rho' = \rho\cos\beta,\ NSM = l - L,\ NEM = \lambda - L;$$

$$\therefore \tan(\lambda - L) = \frac{NM}{EM} = \frac{SN.\sin NSM}{SN.\cos NSM - SE} = \frac{r'.\sin(l - L)}{r'\cos(l - L) - R},$$

$$\rho' = r'.\frac{\sin NSE}{\sin NES} = r'\frac{\sin(l - L)}{\sin(\lambda - L)},$$

$$\tan\beta = \frac{PN}{EN} = \frac{r'\tan b}{\rho'}.$$

From these three equations may be obtained successively λ, ρ', and β; since l, r' and b are given, and L and R are known from the time.

341. If perpendiculars Ee, Nn, are let fall on $E\Upsilon$, $S\Upsilon$, we have

$$r'\cos l - R\cos L = Sn - Se = Eo = \rho'\cos\lambda,$$

$$r'\sin l - R\sin L = Nn - Ee = No = \rho'\sin\lambda.$$

342. *Having given the geocentric, to find the heliocentric place of a planet.*

Here the geocentric latitude and longitude and the position of the Earth, are given, to find the heliocentric latitude and longitude, and the distance of the planet from the Sun.

Let S be the Sun, (fig. 52.) E the Earth, P a planet in its orbit, $S☊$ the line of nodes and axis of x, PN perpendicular to the plane of the ecliptic or plane of xy; BIC, bac spherical triangles situated on surfaces described with centers S and E and radius = 1, and having their angular points determined by the lines SP, SN, $S☊$, EP, EN, ES.

Then BI is the argument of latitude $= u$,

$$\angle BIC = \text{inclination of planet's orbit} = I,$$

$$AI = \Upsilon SE - \Upsilon S☊ = L - n, \quad ac = \Upsilon EN - \Upsilon SE = \lambda - L,$$

$$EM☊ = AI + ac = \lambda - n.$$

Now $\cot BI \sin AI - \cos I \cos AI = \sin I \cot BAI$

$$= -\sin I \cot bac = -\sin I \sin ac \cot bc;$$

$$\therefore \cot u = \frac{\cos I \cos (L - n) - \sin I \cot \beta \sin (\lambda - L)}{\sin (L - n)}.$$

Also $ES \sin AI + EN \sin NM☊ = Nm = PN \cot I$,

or $R \sin (L - n) + \rho \cos \beta \sin (\lambda - n) = \rho \sin \beta \cot I$;

$$\therefore \rho = \frac{R \sin (L - n)}{\sin \beta \cot I - \cos \beta \sin (\lambda - n)}.$$

Hence, u and ρ being determined, we have

$$\sin BC = \sin BI \sin I, \text{ or } \sin b = \sin u \sin I,$$

$$\cos I = \cot BI \tan IC, \text{ or } \tan (l - n) = \tan u \cos I,$$

$$r \sin b = \rho \sin \beta, \text{ or } r = \frac{\rho \sin \beta}{\sin u \sin I}.$$

Also from the triangle BIC, $\cos u = \cos b \,.\, \cos (l - n)$,

and $\sin (l - n) = \cot I \tan b$.

343. Let x, y, z, be co-ordinates of P, x_1, y_1, those of the Earth; then $\tan \beta = \dfrac{z \cos (\lambda - n)}{x - x_1} = \dfrac{z \sin (\lambda - n)}{y - y_1}$; and $\tan (\lambda - n) = \dfrac{y - y_1}{x - x_1}$.

If $S\Upsilon$ the line of equinoxes, be taken for the axis of x, the expressions for x, y, z, are (letting fall perpendiculars from E and N on $S\Upsilon$)

$$x = \rho \cos\beta \cos\lambda + R\cos L,$$
$$y = \rho\cos\beta\sin\lambda + R\sin L,$$
$$z = \rho\sin\beta.$$

Elements of a planet's orbit.

344. The equations for determining the heliocentric place of a planet from the geocentric, contain the values of n, I, &c. depending upon the position of the planet's orbit, of which they require the previous determination. The elements of a planet's orbit are, 1. The inclination of the orbit to the ecliptic. 2. The longitude of the ascending node. 3. The longitude of the perihelion. 4. The mean distance. 5. The eccentricity. 6. The epoch of the planet's being in the perihelion.

The two first elements determine the plane of the body's motion, the third the position of the axis of the orbit, the 4th and 5th the form of the orbit, and the 6th the position of the body at a given time. The period may be substituted for the mean distance, since $P = a^{\frac{3}{2}}$, P being expressed in years, and a in parts of the Earth's mean distance.

345. From the observed geocentric right ascension and declination of a planet at any time, the geocentric latitude and longitude may be found, from which the corresponding values of u and r may be obtained in terms of the elements, by the equations of Art. 342. But the values of r and u (= the longitude of the perihelion + the true anomaly − the longitude of the node) at the time of observation, may be found in terms of the same elements, by the methods of the preceding Chapter; for, the epoch of the body's being in perihelion being assumed, and the time of observation known, the time since the body was in perihelion, i. e. the mean anomaly, and thence the true, and therefore u and r, are found in terms of known and assumed quantities. Therefore by equating values thus found, we obtain two equations for determining the elements. From two other observations four more equations may be obtained; so that three complete observations are sufficient for determin-

ing the six elements of a planet's path. If the path be a circle, the eccentricity and place of the perihelion are not required, and two observations are sufficient for determining the four elements. If the path be a parabola, since the major axis is not required, the three observations furnish one equation more than necessary.

The above, although a theoretical, is not a practical method of finding the elements of a planet's orbit, as the deduced equations, even for the parabola, are much too involved to admit of being solved, and of course would be much more so for the ellipse; so that the direct solution of the problem may be considered impossible. Even the indirect modes which Astronomers have adopted for solving the general problem, are of very considerable difficulty, and not adapted to an elementary treatise like the present. We shall therefore confine ourselves to the case of a circular and a parabolic orbit*.

346. *To find the circular orbit of a planet from two observations.*

Let a be the radius of the orbit, x, y, z co-ordinates of the planet at the time of the first observation, the Sun being the origin, and the axis of x passing through the first point of Aries; also let

$$z + py + qx = 0 \;\ldots\ldots (1),$$

$$z^2 + x^2 + y^2 = a^2 \ldots\ldots (2),$$

be the equations to the circular orbit, and let ρ, β, λ, &c. represent the same quantities as before, of which β, λ, L, R are known from observation; then substituting in equations (1) and (2) the values of x, y, z given in Art. 343, we get

$$\rho\,(\sin\beta + p\cos\beta\sin\lambda + q\cos\beta\cos\lambda)$$

$$+ pR\sin L + qR\cos L = 0,$$

$$\rho^2 + 2R\cos\beta\,.\,\cos(L-\lambda)\,.\,\rho + R^2 = a^2\,;$$

or, calling the known coefficients, A, B, C, D, E, and dividing by $\sin\beta$,

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* See Lagrange, *Mechanique Analytique*, Vol. II. Gauss *Theoria motus corporum celestium*.

$$\rho \cdot (1 + Ap + Bq) + Cp + Dq = 0 \ldots\ldots (3),$$

$$\rho^2 + 2E\rho + R^2 = a^2 \ldots\ldots (4);$$

$$\therefore \rho = \sqrt{a^2 + E^2 - R^2} - E \ldots\ldots (5),$$

by the solution of (4).

Similarly, if ρ', E' be the values of ρ and E at a second observation,

$$\rho' = \sqrt{a^2 + E'^2 - R'^2} - E' \ldots\ldots (6).$$

But if c be the chord joining the extremities of ρ and ρ',

$$\begin{aligned} c^2 &= (x' - x)^2 + (y' - y)^2 + (z' - z)^2 \\ &= 2a^2 - 2 \cdot (xx' + yy' + zz') \\ &= 2a^2 - 2 \cdot \{F\rho\rho' + G\rho R' + H\rho' R + KRR'\} \ldots\ldots (7), \end{aligned}$$

substituting for x, y, z, x', y', z', their values, and calling the known coefficients, F, G, H, K.

Also equating the two values of the circular sector described round the Sun, whilst the planet moves from the one position to the other,

$$a^2 \cdot \sin^{-1}\left(\frac{c}{2a}\right) = \pi a^2 \frac{\text{observed interval}}{\text{planet's period}} = \pi a^2 \frac{t}{a^{\frac{3}{2}}},$$

a being expressed in parts of the radius of the Earth's orbit, and t in parts of a year;

$$\therefore t = \frac{a^{\frac{3}{2}}}{\pi} \cdot \sin^{-1}\left(\frac{c}{2a}\right) \ldots\ldots\ldots\ldots (8).$$

From equations (5) and (6), with an assumed value of a, let ρ and ρ' be calculated, and thence c by equation (7); then this value of c ought, when substituted in (8), to give the observed time (t), if a is rightly assumed. If a is not rightly assumed, the equation (8) will point out whether the error is in excess or defect, and calling α the correction of a, we have, [differentiating the equations (5) and (6)] $\frac{a}{\rho + E} \cdot \alpha$, and $\frac{a}{\rho' + E'} \cdot \alpha$, for the corrections of ρ and ρ', with which we may proceed as before, till a is found to any required degree of accuracy. When a is once found, ρ and ρ' are known from equations (5) and (6); and substituting them in equation (3), and in the equation similarly formed with ρ', two simple equations arise for determining p and q, whence the position of the orbit is known; for, n being the longitude of the node, and I the inclination, since $py + qx = 0$ is the equation to the line of nodes,

$$\tan n = -\frac{q}{p}, \text{ and } \sec I = \sqrt{1 + p^2 + q^2}.$$

Since all the planetary orbits have small eccentricities, the above mode may be used as an approximate method of determining the mean distance, the place of the node, and the inclination of any planetary orbit. When applied to that of Vesta, it gave the inclination within 2′, and the log. of the mean distance within .0056 of its true value. We proceed next to find the approximate elements of a comet's orbit.

Approximate elements of a comet's orbit found from three observations.

347. When a comet appears, the observations made for ascertaining its orbit are of its declinations and right ascensions from which its geocentric latitudes and longitudes are obtained; these observations are made either with the Equatoreal, or by micrometrical measurements of the differences of the right ascension and declination of the comet and a neighbouring known fixed star; and cannot be made at any point of the orbit as in the case of the planets, but only near the perihelia as it is then only the comets are visible. A very eccentric ellipse or hyperbola and a parabola having a common vertex and focus, sensibly coincide for a small space on each side of the vertex; therefore it is difficult to ascertain from observations made of necessity near the perihelion the species of conic section described by the comet; probably all comets move in very eccentric ellipses, returning to our system after very long intervals; if any have hyperbolic orbits, as has been thought, then they can become visible to us but once; but it is usual to assume that comets move in parabolas, as it has in general been found that these agree best with observations.

348. If a comet be observed at three different times, separated by short intervals, its present position in space and future path in the heavens during the same apparition can be assigned, but not its period; and if the parabolic orbit be determined with all possible accuracy, it may be impossible to recognise a comet on its return, by reason of the disturbance experienced in passing near any of the large planets. Of more than a hundred comets whose motions have been computed, the periods of three only have been ascertained, viz. the comet of Halley which has a period of about 76 years and returned to its perihelion in the beginning of November 1835; the comet of Biela which has a period of $6\frac{3}{4}$ years and whose last perihelion passage happened in 1838; and the comet of Encke which has a period of 1210 days or about $3\frac{1}{3}$ years. No comet has hitherto sensibly disturbed the motions of any of the planets: that of 1770 seems to have traversed the system of Jupiter's satellites without producing the slightest alteration. We need not therefore be under apprehensions of having our Astronomical Tables deranged by the near approach of comets, to any body of our system; as none of these move in the plane of the ecliptic, the chance of the Earth coming in collision with any of them is still smaller, and even then the mischief might be only local. These bodies derive interest at present from the probability that they will conduce to the more accurate determination of the mass of Uranus and some other of the planets, and decide the question of the existence of an ether or fluid of small density pervading space. The mean distance of Encke's comet from the Sun is obtained smaller from the later revolutions than from the earlier, which is explained by the hypothesis of a resisting medium, or something which produces almost exactly the same effects.

349. *From three geocentric observations of a comet to find its heliocentric and geocentric distances at the instants of the observations.*

In order to simplify the problem, we shall assume that the Earth moves uniformly in a circular orbit; and that the intervals between the observations are equal, and so small that the arc of the comet's orbit included between the extreme places may be considered as coinciding with its chord, which will consequently be bisected by the comet's place at the second observation.

Let the Sun be the origin of co-ordinates, the ecliptic the plane of xy, and the line of equinoxes the axis of x; r, x, y, z the comet's heliocentric distance and co-ordinates at the first observation,

$$r',\ x',\ y',\ z',\ldots\ldots\ldots\ldots\ldots\text{after a time}\quad t,$$

$$r'',\ x'',\ y'',\ z'',\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 2t;$$

$$\therefore\ 2x' = x + x'',\ \ 2y' = y + y'',\ \ 2z' = z + z'' \quad (1).$$

Let R be the radius of the Earth's orbit, L, L', L'', the heliocentric longitudes of the Earth; λ, λ', λ''; β, β', β'', the geocentric longitudes and latitudes of the comet, and ρ, ρ', ρ'', its curtate distances from the Earth at the times of the several observations; then (Art. 343)

$$x = R\cos L + \rho\cos\lambda,\ y = R\sin L + \rho\sin\lambda,\ z = \rho\tan\beta;$$

and similarly for x', y', z', &c. (2).

Hence, by substituting these values, equations (1) become

$$2\,(R\cos L' + \rho'\cos\lambda') = R\,(\cos L + \cos L'') + \rho\cos\lambda + \rho''\cos\lambda'', \quad (3)$$

$$2\,(R\sin L' + \rho'\sin\lambda') = R\,(\sin L + \sin L'') + \rho\sin\lambda + \rho''\sin\lambda'', \quad (4)$$

$$2\,\rho'\tan\beta' = \rho\tan\beta + \rho''\tan\beta''; \quad (5)$$

therefore, eliminating ρ' between equations (3) and (5),

$$(\rho\cos\lambda + \rho''\cos\lambda'')\tan\beta' - (\rho\tan\beta + \rho''\tan\beta'')\cos\lambda'$$
$$= R\tan\beta'\,\{\cos L' - \cos L'' - (\cos L - \cos L')\}$$
$$= 2\,R\tan\beta'\sin\tfrac{1}{2}\,(L' - L)\ \{\sin\tfrac{1}{2}\,(L' + L'') - \sin\tfrac{1}{2}\,(L + L')\}$$
$$= 4\,R\tan\beta'\sin^2\tfrac{1}{2}\,(L' - L)\cos L' \quad (6);$$

$$\because\ L'' - L' = L' - L,\ \text{since the intervals are equal.}$$

Similarly, the elimination of ρ' between (4) and (5) must give

$$(\rho\sin\lambda + p''\sin\lambda'')\tan\beta' - (\rho\tan\beta + \rho''\tan\beta'')\sin\lambda'$$
$$= 4\,R\tan\beta'\sin^2\tfrac{1}{2}\,(L' - L)\sin L' \quad (7).$$

Hence, eliminating R between (6) and (7),

$$(\rho\cos\lambda+\rho''\cos\lambda'')\tan\beta'\sin L'-(\rho\tan\beta+\rho''\tan\beta'')\cos\lambda'\sin L'$$
$$=(\rho\sin\lambda+\rho''\sin\lambda'')\tan\beta'\cos L'-(\rho\tan\beta+\rho''\tan\beta'')\sin\lambda'\cos L',$$

$$\text{or } \rho\sin(L'-\lambda)\tan\beta'-\rho\sin(L'-\lambda')\tan\beta$$
$$=\rho''\sin(L'-\lambda')\tan\beta''-\rho''\sin(L'-\lambda'')\tan\beta';$$

$$\therefore\ \rho=\frac{\sin(L'-\lambda')\tan\beta''-\sin(L'-\lambda'')\tan\beta'}{\sin(L'-\lambda)\tan\beta'-\sin(L'-\lambda')\tan\beta}\cdot\rho''=\frac{1}{m}\rho''$$ suppose, m being, as we see, a function of known quantities.

Now

$$r^2=x^2+y^2+z^2=\rho^2\sec^2\beta+2\rho R\cos(\lambda-L)+R^2,$$
$$r''^2=m^2\rho^2\sec^2\beta''+2m\rho R\cos(\lambda''-L'')+R^2,$$
$$c^2=(x''-x)^2+(y''-y)^2+(z''-z)^2=r^2+r''^2-2(xx''+yy''+zz'')$$
$$=r^2+r''^2-2m\rho^2\{\cos(\lambda-\lambda'')+\tan\beta\tan\beta''\}$$
$$-2R\rho\{\cos(\lambda-L'')+m\cos(\lambda''-L)\}-2R^2\cos(L-L''),$$

c being the chord of the arc included between the extreme places; or, denoting the known coefficients by A, B, C, &c.

$$r^2=A+B\rho+C\rho^2,$$
$$r''^2=A''+B''\rho+C''\rho^2,$$
$$c^2=F+G\rho+H\rho^2.$$

From these equations with any assumed value of ρ, let r, r'', c be calculated, then, if the assumption be correct, the equation

$$2t=\frac{365\frac{1}{4}}{12\pi}\cdot\{(r+r''+c)^{\frac{3}{2}}-(r+r''-c)^{\frac{3}{2}}\}\ldots\ldots(8),$$

(Art. 324.) must be satisfied.

If this equation is not satisfied, let $\rho+\Delta\rho$ be a corrected value of ρ, then

$$r + \frac{B + 2C\rho}{2r} \cdot \Delta\rho, \qquad r'' + \frac{B'' + 2C''\rho}{2r''} \cdot \Delta\rho,$$

$$\text{and } c + \frac{G + 2H\rho}{2c} \cdot \Delta\rho,$$

are the corrected values of r, r'' and c, which must be proceeded with as before, till values of r, r'', c are found, which satisfy equation (8).

350. If we eliminate ρ'' between equations (6) and (7), we find

$$\rho = \frac{4R \sin^2 \frac{1}{2}(L' - L) \{\sin (L' - \lambda'') \tan \beta' - \sin (L' - \lambda') \tan \beta''\}}{\tan \beta \sin (\lambda'' - \lambda') + \tan \beta'' \sin (\lambda' - \lambda) - \tan \beta' \sin (\lambda'' - \lambda)}$$

which gives the curtate distance at once in known quantities; but the denominator is very small and greatly influenced by the errors of observation; in fact, if the three places of the comet are in the same apparent great circle, the denominator, by a well known theorem, becomes equal to zero.

We now proceed to apply our results to the determination of the elements of the orbit.

351. *From the quantities determined in the preceding article, that is, the curtate distance from the Earth and distance from the Sun at two observations, to find the inclination, and the place of the node of a comet's orbit, and the argument of latitude at each of the observations.*

Let l, l'' be the heliocentric longitudes of the comet at the first and third observation, A and A'' the geocentric longitudes of the Sun, which are $= 180^0 + L$ and $180^0 + L''$; then (fig. 53.)

$$\sin NSE = \frac{EN}{SN} \cdot \sin SEN = \frac{\rho \cdot \sin (A - \lambda)}{r \cos b}.$$

In this equation A and λ are known from observation, ρ and r have been determined in the preceding article, and b the comet's heliocentric latitude is known from the equation $r \sin b = \rho \tan \beta$. Hence NSE is known, and adding to this $ME♈$ the heliocentric longitude of the Earth, the heliocentric *reduced* longitude of the comet (l) is obtained. Similarly, from ρ'', r'', may l'', b'', the heliocentric longitude and latitude at the last observation, be found. Let I be the inclination of the comet's orbit, n the longitude of the node; then (Art. 342)

$$\sin(l-n) = \tan b \,.\, \cot I$$

$$\sin(l''-n) = \tan b'' \,.\, \cot I;$$

$$\therefore \frac{\sin(l-n)+\sin(l''-n)}{\sin(l-n)-\sin(l''-n)} = \frac{\tan b + \tan b''}{\tan b - \tan b''},$$

$$\text{or } \frac{\tan\left\{\frac{1}{2}(l+l'')-n\right\}}{\tan\frac{1}{2}(l-l'')} = \frac{\sin(b+b'')}{\sin(b-b'')};$$

$$\therefore \tan\left\{\tfrac{1}{2}(l+l'')-n\right\} = \frac{\sin(b+b'')}{\sin(b-b'')} \,.\, \tan\tfrac{1}{2}(l-l''),$$

from which equation n may be found; and therefore I, and the arguments of latitude at the first and last observations u, u'', from the equations (Art. 342.)

$$\cot I = \sin(l-n) \,.\, \cot b,$$

$$\cos u = \cos b \,.\, \cos(l-n),$$

$$\cos u'' = \cos b'' \cos(l''-n).$$

352. *Given two arguments of latitude in a parabolic orbit, and the corresponding distances from the Sun, and the longitude of the node, to find the true anomaly, the longitude of the perihelion, the perihelion distance, and time of perihelion passage.*

Let $4a$ be the parameter of the orbit, π the distance of the perihelion from the node, and u, u'', two arguments of latitude determined as in the preceding article; then $u-\pi$ and $u''-\pi$ are the true anomalies corresponding to the distances r and r'', and if $u-\pi=v$, then $u''-\pi = v+u''-u$;

$$\therefore \cos\tfrac{1}{2}v = \sqrt{\frac{a}{r}}, \quad \cos\tfrac{1}{2}\left\{v+(u''-u)\right\} = \sqrt{\frac{a}{r''}};$$

$$\therefore \cos\tfrac{1}{2}(u''-u) - \sin\tfrac{1}{2}(u''-u) \,.\, \tan\tfrac{1}{2}v = \sqrt{\frac{r}{r''}}, \text{ by division;}$$

$$\therefore \tan \tfrac{1}{2} v = \frac{\sqrt{r''} . \cos \frac{1}{2} (u'' - u) - \sqrt{r}}{\sin \frac{1}{2} (u'' - u) \sqrt{r''}}, \text{ which gives } v,$$

and therefore $\pi = u - v$, is known, and the longitude of the perihelion $n + \pi$.

The perihelion distance $a = r \cos^2 \frac{1}{2} v$ is consequently known. Let the heliocentric longitude of the perihelion be less than that of the comet at the first observation, and consequently the perihelion passage precede the observations; then at the instant of the first observation the time elapsed since the perihelion passage (Art. 323)

$$= \frac{a^{\frac{3}{2}}}{\pi \sqrt{2}} \left\{ \tan \frac{v}{2} + \tfrac{1}{3} \tan^3 \frac{v}{2} \right\}.$$

Since $\tan \frac{1}{2} v$ contains only r, r'' and $u'' - u$, the true anomaly in a parabolic orbit may be found from two distances, and the angle between.

Since the radius vector in a parabola does not divide the chord accurately as the times, the above mode of finding the elements of a comet's orbit must be incorrect; the error is, however, very small, and the mode of correcting it may be seen in the Essay of Olbers, whence the preceding Articles are extracted.

Elements of a planet's orbit found from particular observations.

353. The elements of a planet's orbit may be obtained much more simply from observations made under favourable circumstances, than by the general methods alluded to in Art. 345, which have been applied to the cases of circular or parabolic orbits; and although in the case of a new planet being discovered, the elements would probably be determined to a considerable degree of accuracy long before such circumstances would present themselves, yet the following methods are worthy of attention as affording the best means of correcting the nearly known elements.

354. *To find the place of the node of a planet's orbit from observations made on the planet in its node.*

Let S (fig. 55.) be the Sun, E the Earth, and P the planet when its geocentric latitude, calculated from the observed declination and right ascension, is 0, and therefore in its node.

Let $SE = R$, $SP = r$, L the heliocentric longitude of the Earth, l of the planet in its node, therefore $ESP = L - l$; and let $E = SEP$ the elongation of the planet from the Sun. Then

$$\frac{r}{R} = \frac{\sin SEP}{\sin SPE},$$

$$\text{or } r \,.\, \sin (E + S) = R \,.\, \sin E,$$

$$\therefore r \,.\, \sin (E + L - l) = R \,.\, \sin E.$$

Let E', L', R', be values of E, L, and R, when the planet returns to the node after one or more revolutions; therefore,

$$r \,.\, \sin (E' + L' - l) = R' \,.\, \sin E' \ldots\ldots\ldots\ldots (2);$$

$$\therefore \frac{\sin (E' + L' - l) + \sin (E + L - l)}{\sin (E' + L' - l) - \sin (E + L - l)} = \frac{R' \sin E' + R \sin E}{R' \sin E' - R \sin E};$$

$$\therefore \frac{\tan \left\{\frac{1}{2}(E' + E + L' + L) - l\right\}}{\tan \frac{1}{2}(E' - E + L' - L)} = \frac{R' \sin E' + R \sin E}{R' \sin E' - R \sin E};$$

$$\therefore \tan \left\{\tfrac{1}{2}(E' + E + L' + L) - l\right\}$$

$$= \frac{R' \sin E' + R \sin E}{R' \sin E' - R \sin E} \,.\, \tan \tfrac{1}{2}(E' - E + L' - L),$$

which determines l the longitude of the node.

355. If no observed right ascension and declination give the planet's geocentric latitude 0, the time when the planet is in its node, which is requisite for finding the values of L and E, must be found by a proportion. Let β and β' be the geocentric latitudes of the planet on successive days before and after passing the node, then, $\beta + \beta' : \beta ::$ time between the two observations : the time between the planet's being in the node and the first observation; which, added to the time of the first observation gives the time when the planet is in the node; for which time the values of E and L must be found in the same manner by taking the proportional parts.

This method will be slightly incorrect on account of the slow motion of the nodes, which is not noticed; but if the node be determined in this way at different times, and the motion of the node be thence deduced, the angle described by the node between the two observations may be found; and if this be called a, and the value $l + a$ be used instead of l in equation (2), the place of the node may be found more accurately.

356. *To find the inclination of a planet's orbit, from observations made on the planet when the Earth is in the line of Nodes.*

Having found the longitude of the node by the last Article, let E (fig. 56.) be the Earth in the line of nodes, S the Sun, P the planet, $P\Pi$ perpendicular to the ecliptic.

Let, as before, $SE\Pi$ the difference of geocentric longitudes of S and $P = E$, $PE\Pi$ the geocentric latitude $= \beta$, and $I =$ the inclination of the orbit; also let abc be a spherical triangle on the surface of a sphere described about E with radius 1; therefore, by Napier's rules, since the angle at c is 90^0,

$$\sin ac = \tan bc \cot cab, \text{ or } \sin E = \tan\beta . \cot I;$$

$$\therefore \cot I = \sin E . \cot\beta.$$

357. *To find when a planet is in conjunction with the Sun.*

When a planet is in conjunction, its geocentric longitude is the same as that of the Sun.

Let E and E' be the elongations of the planet's place referred to the ecliptic from the Sun, T and T' the corresponding times on successive days before and after conjunction, and C the time of conjunction, all reckoned from the same epoch.

Then, considering the change of elongation uniform for the time $T' - T$,

$$\frac{C - T}{T' - T} = \frac{E}{E' + E};$$

$$\therefore C = T + \frac{E}{E' + E} . (T' - T).$$

358. If β and β' be the geocentric latitudes at the times T and T' and $\beta + \Delta\beta$ at conjunction,

$$\frac{\Delta\beta}{\beta'-\beta}=\frac{C-T}{T'-T}=\frac{E}{E'+E},$$

$$\therefore \Delta\beta=\frac{E}{E'+E}\cdot(\beta'-\beta).$$

Therefore, the geocentric latitude at conjunction

$$=\beta+\frac{E}{E'+E}\cdot(\beta'-\beta).$$

359. The heliocentric longitude of the planet at conjunction is the same as that of the Earth, and may be found in exactly the same manner as the geocentric latitude. If L and L' be the two longitudes of the Earth on successive days before and after conjunction, the heliocentric longitude of the Earth at conjunction or, which is the same thing, the heliocentric longitude of the planet

$$=L+\frac{E}{E'+E}\cdot(L'-L).$$

In exactly the same manner may be found the time when a superior planet is in *opposition to the Sun;* i. e. when the difference between the geocentric longitudes of the planet and the Sun is 180°.

Hence may be obtained the *synodic* time of a planet, which is the interval between two succeeding similar conjunctions. If the two similar conjunctions are taken, separated by a long period, the whole time between the conjunctions divided by the number of revolutions made in the interval, will give the mean synodic time more accurately.

360. *To find a planet's distance from the Sun, and the argument of latitude, at the time of conjunction.*

Let r (fig. 57.) be the distance SP of a planet from the Sun at the time of conjunction,

b the heliocentric latitude PSE,
β the geocentric .... $PE\Pi$,
u the argument of latitude NSP.

Let nSN be the line of nodes, and let $SE = R$; then NSE = the heliocentric longitude of the Earth – that of the node $= L - n$. Therefore, abc being a spherical triangle on the surface of a sphere described about S with radius 1, we have by Napier's rules, since the angle at c is 90^0,

$$\sin cb = \tan ac \cot abc, \text{ or } \tan b = \sin (L - n) \tan I,$$

$$\cos abc = \cot ab \tan bc, \text{ or } \cot u = \cot (L - n) \cos I,$$

which determine b and u;

$$\text{and } r = R \frac{\sin \beta}{\sin (b + \beta)}.$$

361. *Given three distances of a planet from the Sun, and the corresponding arguments of latitude, to find the place of the perihelion and the true anomaly at the first observation.*

Let u, $u + a$, $u + \beta$, be the three arguments of latitude, r, r', r'', the corresponding distances from the Sun, determined by observations made at three different conjunctions, as in the last Article; then, if v be the true anomaly, corresponding to the distance r, $v = u - \pi$, π being the distance of the perihelion from the node. The true anomalies at the second and third observations are $v + a$, $v + \beta$; therefore,

$$r . (1 + e \cos v) = a . (1 - e^2) \ldots\ldots\ldots (1),$$

$$r' . \{1 + e \cos (v + a)\} = a . (1 - e^2) \ldots\ldots\ldots (2),$$

$$r'' . \{1 + e \cos (v + \beta)\} = a . (1 - e^2) \ldots\ldots\ldots (3);$$

$$\therefore r' - r + e . \{r' \cos (v + a) - r \cos v\} = 0, \text{ from (1) and (2)},$$

$$r'' - r + e . \{r'' \cos (v + \beta) - r \cos v\} = 0, \text{ from (1) and (3)}.$$

Therefore, eliminating e between these equations,

$$(r' - r) . \{r'' \cos (v + \beta) - r \cos v\} = (r'' - r) \{r' \cos (v + a) - r \cos v\};$$

therefore, expanding $\cos (v + a)$ and $\cos (v + \beta)$, and dividing by $\cos v$,

$$(r' - r) . \{r'' \cos \beta - r'' \sin \beta . \tan v - r\}$$

$$= (r'' - r) . \{r' \cos a - r' \sin a \tan v - r\};$$

$$\therefore \tan v = \frac{(r' - r'')r + (r'' - r)\, r' \cos \alpha - (r' - r)\, r'' \cos \beta}{(r'' - r)\, r' \sin \alpha - (r' - r)\, r'' \sin \beta},$$

which determines v.

$$\text{Therefore, } e = \frac{r - r'}{r' \cos(v + \alpha) - r \cos v},$$

$$a = \frac{r}{1 - e^2} \cdot (1 + e \cos v),$$

and the longitude of the perihelion $(u - v + n)$, are determined.

The values of v, e, and a, contain only the three distances and the angles α and β, and may, therefore, be found from these data; but the argument of latitude u is requisite for determining the place of the perihelion.

362. *To find a planet's period.*

Let the time when the planet is in its node be observed, then, if the same observations are made after one or more revolutions of the planet, the whole time divided by the number of revolutions is the period with respect to the node; this is not the sidereal period, or the time from a fixed star till the planet returns to it again, on account of the regression of the node on the orbit between the observations. If this regression, calculated as in Art. 355, be a, and n be the number of revolutions, $n \,.\, 360^0 - a$ is the whole angle described by the planet;

$$\therefore \text{the sidereal periodic time} = (\text{the observed time}) \frac{360^0}{n \,.\, 360^0 - a}.$$

363. The period may be also found from the synodic time; for if this time be called S, and P and p be the sidereal periodic times of the Earth and planet all expressed in days; $\frac{360^0}{p}$ and $\frac{360^0}{P}$ are the mean daily motions; therefore $\frac{360^0}{p} - \frac{360^0}{P}$ is the mean daily separation of the Earth and planet, which in the time S ought to be 360^0;

$$\therefore \left(\frac{360^0}{p} - \frac{360^0}{P}\right) S = 360^0;$$

$$\therefore \frac{1}{p} - \frac{1}{P} = \frac{1}{S}; \quad \therefore p = \frac{PS}{P+S}.$$

p is here supposed less than P, or the planet an inferior one; if the planet be superior, it may be proved in the same way that $p = \dfrac{PS}{S-P}$.

The period might have been obtained from the mean distance a, found in the preceding Articles, by the proportion P : 1 year :: $a^{\frac{3}{2}}$: (rad. of Earth's orbit)$^{\frac{3}{2}}$; but, on account of the errors to which the determination of a is liable, P can be found more accurately by the method here given, and then the above proportion will serve to determine a.

The periods round the Sun of the inferior planets are observed to be less, and the periods of the superior planets, greater than a year, as we should expect, since the period varies as (mean dist.)$^{\frac{3}{2}}$.

364. As none of the preceding methods are exact, the elements of the orbits of the planets, so determined, will be subject to slight errors, and will require subsequent corrections, which must be applied by comparing the observed places with the places calculated from the nearly known elements, and thus determining the errors in geocentric latitude and longitude. If the equations connecting the geocentric and heliocentric places of a planet are differentiated, the errors in the heliocentric place, corresponding to the errors in the geocentric place, are obtained; and again, by differentiating the equations by which the heliocentric place depends on the elements, the errors in the heliocentric place are obtained in terms of the errors in the elements. Thus an equation is obtained connecting the observed errors in the geocentric place with the errors of the elements, and from as many equations of this kind as there are errors, the errors may be found. This is a brief outline of the method which is attended with considerable difficulty in the details. See Littrow, Vol. II. p. 201.

Points of station, and retrograde motions of the Planets.

365. *To find the geocentric motion of a planet, supposing the Earth and planet to describe circles in the ecliptic.*

Let a and ρ be the distances of a planet from the Sun and Earth, the Earth's distance from the Sun being 1. Then, l and L being as before the heliocentric longitudes of the planet and Earth, and λ the geocentric longitude of the planet, and t the time from known positions of the Earth and planet,

$$a \cos l - \cos L = \rho \cos \lambda, \quad \text{Art. 341},$$

$$a \sin l - \sin L = \rho \sin \lambda;$$

$$\therefore \tan \lambda = \frac{a \sin l - \sin L}{a \cos l - \cos L}; \qquad \therefore \frac{d_t\lambda}{\cos^2\lambda}$$

$$= \frac{(a\cos l - \cos L).(a\cos l.d_tl - \cos L.d_tL) + (a\sin l - \sin L)(a\sin l.d_tl - \sin L.d_tL)}{(a\cos l - \cos L)^2}$$

$$= \frac{\{a^2 - a\cos(L-l)\}\, d_tl + \{1 - a.\cos(L-l)\}\, d_tL}{(a\cos l - \cos L)^2}.$$

But $d_tL : d_tl :: a^{\frac{3}{2}} : 1$; $\therefore d_tL = a^{\frac{3}{2}}.d_tl$;

therefore, substituting this value, and making

$$\frac{\cos\lambda}{a\cos l - \cos L} = P,$$

$$d_t\lambda = P^2.\{a^2 + a^{\frac{3}{2}} - (a + a^{\frac{5}{2}}).\cos(L-l)\}\, d_tl.$$

366. When an inferior planet is in the conjunction nearest the Earth, or a superior planet in opposition, $L - l = 0$,

$$\therefore d_t\lambda = P^2a.(a + a^{\frac{1}{2}} - 1 - a^{\frac{3}{2}}).d_tl$$

$$= P^2a(a-1).(1 - a^{\frac{1}{2}}).d_tl,$$

which is negative, whether a be greater or less than 1; that is, the planet's apparent motion is retrograde.

When a superior planet is in conjunction, or an inferior planet in the conjunction farthest from the Earth, $L - l = 180^\circ$, and $\cos(L-l) = -1$; therefore $d_t\lambda$ is positive, or the apparent motion is direct. Hence the apparent motions of all planets are direct in apogean syzygy, and retrograde in perigean syzygy.

367. When $d_t\lambda = 0$, the planet appears stationary. In this case

$$\cos(L-l) = \cos ESP = \frac{a + a^{\frac{1}{2}}}{1 + a^{\frac{3}{2}}} = \frac{1}{a^{\frac{1}{2}} + a^{-\frac{1}{2}} - 1} = \frac{1}{1 + (a^{\frac{1}{4}} - a^{-\frac{1}{4}})^2},$$

a quantity necessarily less than 1, whether a be less or greater than 1; that is, all planets must sometimes appear stationary. Let α, θ, be the values of $L - l$, (that is, the heliocentric elongation of the planet from the Earth), when the planet is stationary, and at any other time;

$$\therefore\ d_t\lambda = P^2\,(a + a^{\frac{5}{2}})\,(\cos\alpha - \cos\theta)\,d_t l\,;$$

hence in a synodic revolution, that is, between $\theta = 0$ and $\theta = 360^0$, the apparent motion is retrograde from $\theta = 0$ to $\theta = \alpha$, and from $\theta = 360^0 - \alpha$ to $\theta = 360^0$, and direct for all other values of θ. If t be the time from syzygy to the time when the planet is stationary, and m and m' be the daily motions of the planet and the Earth in longitude, and therefore $m \sim m'$ their daily angular separation,

$$\alpha = t\,(m \sim m'), \quad \text{or } t = \frac{\alpha}{m \sim m'}.$$

Therefore a planet's motion appears retrograde whilst it describes an arc of its orbit $= 2ma\alpha \div (m \sim m')$, and progressive whilst it describes the arc $= 2ma\,(\pi - \alpha) \div (m \sim m')$.

If the Earth were at rest, or $d_t L = 0$, we should have

$$d_t\lambda = P^2 . \left\{a^2 - a\cos(L - l)\right\},$$

which could never become zero for a superior planet; therefore, as the superior planets often appear stationary, this affords another presumption that the Earth moves.

368. The above will be better understood by referring to fig. 22, Plate I, where TT_1, PP_1, represent the orbits of the Earth and an inferior planet (Mercury or Venus,) supposed to be circles in the plane of the ecliptic, with the Sun S in their common center. Let P be the planet in inferior conjunction, seen among the fixed stars at p; and in any time let the Earth and planet describe the angles TST_2, PSP_2 in the same direction,

of which the latter is the greater, because the angular velocity varies inversely as the periodic time, that is, inversely as (radius of orbit)$^{\frac{3}{2}}$; then the planet is seen in the direction T_2P_2 produced, at p_2, and therefore in the time from conjunction appears to have retrograded through pp_2. This apparent motion backwards will become slower every day, till the lines joining the Earth and planet on consecutive days are parallel (that is, till the velocities of the bodies resolved perpendicular to the line joining them are equal,) when the planet will seem to occupy the same point among the fixed stars on consecutive days, or to be stationary; this will happen nearly when the visual ray is a tangent to the planet's orbit, or when the planet has its greatest elongation; in the figure we will suppose p_2 to represent the point of station. But if we take $TT_1 = TT_2$, $PP_1 = PP_2$, then T_1, P_1, are contemporary positions of the Earth and planet before conjunction, and since the directions of their motions in these positions are inclined at exactly the same angles to the line joining them as when they are at T_2, P_2, the planet must appear stationary at p_1, and will retrograde through p_1p till conjunction; hence there will be two points of station at equal intervals before and after inferior conjunction, and the whole arc of retrogradation will be p_1p_2. As the bodies advance in their orbits from T_2 and P_2, the apparent place of the planet after being stationary at p_2 will retrace its course through p_2p_1, till after half a synodic revolution it is in superior conjunction at p_3, P_3 and T_3 being the corresponding places of the planet and Earth. After leaving p_3 the motion of the apparent place will continue to be direct till (p_5 being the point of inferior conjunction,) it has reached a point p_4 beyond p_5, so that $p_4p_5 = pp_1$; it will then retrograde through p_4p_6, and become stationary at p_6, a point at the same distance from p_5 as p_4 is; the apparent motion will then be direct through the following superior conjunction, and the same variations will recur in the same order.

The angle through which Mercury appears to retrograde at every inferior conjunction varies from $8^0.33'$ to $16^0.18'$, and occupies 24 days in the first case, and 20 days in the second; for Venus the angle varies from $15^0.20'$ to $16^0.31'$, and the corresponding times are 41 and 43 days respectively.

369. Again, suppose T to represent a superior planet and P the Earth, then the planet will be seen in opposition among the fixed stars at t, and as the planet describes the arc TT_2, and the Earth the arc PP_2, the apparent place will retrograde through tt_2 and become stationary at t_2; also before opposition it will be stationary at t_1, and in the time to opposition it will retrograde through the arc t_1t; the apparent place will then move directly from t_2 to t_3 where it is in conjunction, (the Earth being at P_3 and the planet at T_3) and will go on in the same direction till, as it approaches opposition, it becomes stationary as at t_4; it will then retrograde through the place of opposition t_5, to an equal distance on the other side t_6, when it will become stationary; after which, its motion will again become direct, and the same alternations recur.

The angle through which Mars appears to retrograde at every opposition varies from $11^0.8'$ to $19^0.30'$, and occupies 62 days in the former case, and 81 in the latter; and his elongation when stationary varies from $129^0.2'$ to $145^0.27'$. For Jupiter and Saturn the angles of retrogradation are 10^0 and $6^0.48'$, the times, 119 days and 137 days, and the angles of elongation when stationary, 115^0 and 109^0 respectively.

370. From the observed elongation of a planet when stationary, to find its distance from the Sun, supposing the Earth and planet to describe circles in the ecliptic.

The planet and Earth, as has been stated, will appear stationary to one another when their velocities resolved perpendicular to the line joining them are equal; for in that case the line joining them, i. e. the direction in which the planet is seen will move parallel to itself; and therefore the planet will apparently occupy, for a short time, the same position among the fixed stars. Now (fig. 55. Plate III.)

the Earth's velocity resolved perpendicular to $PE = v \cos SEP$,

and the planet's.................................... $= v' \cos SPE$,

v and v' being their respective velocities in their orbits in directions perpendicular to SE, SP;

$$\therefore \left(\frac{\cos P}{\cos E}\right)^2 = \frac{v^2}{v'^2} = \frac{SP}{SE} = a, \quad \text{also } \frac{\sin E}{\sin P} = \frac{SP}{SE} = a;$$

$$\therefore \frac{1-\sin^2 P}{\cos^2 E} = \frac{1-\frac{1}{a^2}\sin^2 E}{\cos^2 E} = a,$$

$$\text{or } 1+\tan^2 E - \frac{1}{a^2}\tan^2 E = a,$$

$$\therefore\ a^2 = (1+a)\tan^2 E;$$

and solving the quadratic

$$a = \tfrac{1}{2}\tan^2 E + \tfrac{1}{2}\tan E\sqrt{4+\tan^2 E}.$$

The equation $\tan E = \frac{a}{\sqrt{1+a}}$ gives the planet's elongation from the Sun, when stationary. By this means the mean distance of Pallas was approximated to, as that planet was stationary soon after it was first discovered.

Phases of the Planets.

371. The planets are too distant to present to the naked eye any very obvious change of appearance in their different positions with regard to the Sun; but Venus, Mercury and Mars, which are situated most favourably for such observations, shew on a small scale the same phases as the Moon does, or such as would be shewn by opaque spherical bodies illuminated solely by the rays of the Sun.

372. *To find the phases of an opaque heavenly body illuminated solely by the Sun's rays.*

On that supposition it would be possible to see only one portion of the surface of the body, viz. the portion illuminated, for the rest would transmit no light; and of the visible or enlightened portion, we should see only so much as was situated in the hemisphere turned towards us. To ascertain the shape and dimensions which the part visible to us would have, for different positions of the body in its orbit, let M fig. 18, Plate I. be the center of the body, and TMS a plane passing through it, and the centers T and S of the Earth and Sun, and cutting the body according to the circle $xyon$; then the plane $xzop$ perpendicular to TM cuts off the hemisphere which can transmit light to T, and the plane $nzmp$ perpendicular to SM, the hemisphere which can receive light from S; consequently the zone $xzmp$ is that part of the surface which is visible at T. Let zqp be the projection of zmp, or the ellipse traced out by dropping perpendiculars from every point of the circle zmp upon the plane $xzop$; then it is clear that to a spectator at T, by reason of his great distance, the effect is the same as if the light proceeded from every point of the flat surface $zqpx$; consequently the appearance presented is that of a luminous area, bounded on the side nearest the Sun by a semicircle, and on the opposite side by a semi-ellipse.

But the area zqp = area $zmp \times \cos mMq$,

$\therefore$ apparent visible area $xzqp$ = semicircle $zxp\,(1 + \cos mMq)$;

but $1 + \cos mMq = 1 - \cos mMx = 1 - \cos SMy' = \text{versin}\,SMy'$,

$\therefore$ apparent visible area $= \frac{1}{2}$ real disk × versine of exterior $\angle$ of elongation.

It is evident that the breadth of the phase xq will vary according to the same law, i. e. as the versed sine of the exterior angle of elongation. The exterior boundary of the visible area is always a complete semicircle; however small therefore that area be, the line joining its extreme points or *cusps* is a diameter of the body perpendicular to the plane through its center, and those of the Earth and Sun.

373. If the planet be supposed an inferior planet, and its orbit to coincide with the ecliptic, the exterior angle of elongation is 0 at the nearer conjunction, and increases to 180^0 at the superior conjunction; therefore the illuminated part of the disk is 0 at the nearer conjunction, and increases gradually to the farther conjunction, where it equals the whole disk. As the angle of elongation becomes greater than 180^0, the illuminated part diminishes, till the angle of elongation = 360^0, or the planet is again in conjunction, when no part of the disk is illuminated. The planet's orbit is here supposed to coincide with the ecliptic, but as that is not exactly the case for either of the inferior planets Venus and Mercury, the angle of elongation never vanishes, unless the planet be in its node at the time of conjunction; therefore, except in this case, some portion of the disk is always illuminated.

In the same manner may the phases of a superior planet be found from the above expression for the illuminated part.

On account of her proximity to the Earth, Venus is generally the most conspicuous of the planets, appearing like a bright star in the heavens. When near conjunction, she is above the horizon only during day-light and is of course not visible to the naked eye; and as her greatest elongation from the Sun always lies between 45°. 24′ and 47°. 18′, she can never be visible except at an altitude less than the greater of these limits, in the west or east, for a short period after sunset or before sunrise, being thus a Morning and Evening Star alternately. The same is true for Mercury, whose greatest elongation from the Sun varies from 17°. 50′ to 27°. 42′; but they shine with a variable splendour by reason of changes both in their phases, and especially in their distances from the Earth; the apparent diameters of Mercury and Venus being 12″ and 62″ respectively, at inferior conjunction; and only 4″,4 and 9″,5 at superior conjunction.

In the following synoptical table of the elements of the solar system, the elements of the new planets, Vesta, Juno, Ceres and Pallas, which are not known with great accuracy, are reduced to the epoch of midnight, December 31, 1819, Paris time. All the other quantities in the table are for the epoch of midnight, December 31, 1800, Paris time. The numerical data are chiefly those given by Hansen in Schumacher's Jahrbuch for 1837.

TABLE OF THE SOLAR SYSTEM.

Planet.	Sidereal Period.	Mean Distance.	Eccentricity.
Mercury . .	$87^{d}.9692580$	.3870981	.20551494
Venus. . . .	224 .7007869	.7233316	.00686074
Earth	365 .2563835	1.0000000	.01685318
Mars	686 .9796458	1.5236923	.0933070
Vesta	1325 .7431	2.3678700	.089130
Juno	1592 .6608	2.6690090	.257848
Ceres	1681 .3931	2.7672450	.078439
Pallas. . . .	1686 .5388	2.7728860	.241648
Jupiter. . .	4332 .5841212	5.2027760	.0481621
Saturn . . .	10759 .2198174	9.5387861	.0561505
Uranus. . .	30686 .8208296	19.1823900	.0466108

	Mean Longitude.	Long. Perihelion.	Long. Node.
Mercury . .	163° 56′ 27″	74° 21′ 47″	45° 57′ 31″
Venus. . . .	10 44 35	128 37 1	74 52 40
Earth	100 9 13	99 30 5	. . . .
Mars	64 7 2	332 24 24	48 1 28
Vesta	278 21 57	249 33 24	103 11 30
Juno	200 9 32	53 33 58	171 9 58
Ceres	123 39 41	147 7 31	80 41 24
Pallas. . . .	108 18 17	121 7 4	172 39 27
Jupiter. . .	112 12 36	11 8 35	98 25 34
Saturn . . .	135 20 22	89 8 58	111 35 47
Uranus. . .	177 47 18	167 21 42	72 51 14

TABLE OF THE SOLAR SYSTEM.

Planet.	Inclination.	* Secular Sidereal Variation of		
		Long. of Perihel.	Long. of Node.	Inclination.
Mercury	7° 0′ 0″	+ 581″	−1007″	+18″.4
Venus . .	3 23 25	− 324	−2050	− 7 .2
Earth . .	....	+1125	....	....
Mars . .	1 51 0	+1546	−2522	− 1 . 3
Vesta . .	7 8 9	unknown.	....	....
Juno . .	13 4 45	....	....	....
Ceres . .	10 37 26	....	....	....
Pallas . .	34 34 55	....	....	....
Jupiter .	1 18 52	+ 665	−1590	−23 . 0
Saturn .	2 29 38	+1931	−1954	−15 . 0
Uranus .	0 46 25	+ 228	−3605	+ 3 . 0

	Time of Rotation.	Mass.	Volume.	True Diameters.
Sun . . .	25ᵈ 12ʰ	354936. 000	1407124 . 0	112 . 06
Mercury	24ʰ 5′	0. 1627	0 . 06	0 . 391
Venus . .	23 21	0.88466	0 . 957	0 . 985
Earth . .	23 56	1.0	1 . 0	1 . 0
Mars . .	24 37	0.13242	0 . 14	0 . 519
Jupiter .	9 55	338. 453	1414 . 2	11 . 225
Saturn .	10 29	101. 064	734 . 8	9 . 022
Uranus .	unknown.	1. 690	82 . 0	4 . 344

The apparent equatoreal semi-diameters, for the mean distance of the Earth from the Sun, are Mercury 3″,23, Venus 8″,25, Mars 4″,435, Jupiter 99″,704, Saturn 81″,106, Uranus 37″,25.

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* If to the secular sidereal variations of Long. of Perihel. and Long. of Node the precession in 100^{y}, viz. 5023″ be added, we get the secular tropical variations of those elements.

CHAPTER XV.

THE MOON AND SATELLITES.

374. To a spectator on the Earth's surface, as has been said, the Moon appears to move in the same direction as the Sun, or according to the order of the signs, and by this motion to complete its circuit round the heavens in about twenty-seven days. Its orbit lies within that which the Sun appears to describe round the Earth, because it often passes between the Sun and the Earth, intercepting all or a portion only of the Sun's light, and causing eclipses of that luminary; and its elongation from the Sun passes through all degrees of magnitude from 0^0 to 180^0; these facts shew that the circumstances of its motion are different from those both of the superior and inferior planets; and that in order to produce the observed appearances the Moon must be supposed to revolve round the Earth in a month, while the Earth itself is carried over an arc of its annual orbit round the Sun.

375. Of the actual path of the Moon some idea may be acquired from the following illustration.

From physical astronomy it appears that the Moon's path in space is always concave to the Sun; consequently if TT' fig. 17. Plate I. represent the Earth's orbit about S the Sun, the Moon's path or at least its projection on the ecliptic, will resemble the dotted line MFM', M and M' being consecutive positions of the Moon when in conjunction, and F her place at opposition, and the portion MFM' consequently recurring something more than twelve times. When T and M are the positions of the Earth and Moon, the spectator sees the Moon referred to the heavens at L; when t, m, are their positions, he sees her in the direction tm, or, being unconscious of his own motion, in TL' parallel to tm, i. e. the Moon has apparently described the angle LTL' about him; and the ratio of the distances TM, tm, determined from the apparent diameters, or of TL, TL', which are taken proportional to them, is observed to agree with the supposition of the Moon's describing an ellipse; thus it is that, although the Moon's real path is a complicated curve of double curvature, the illusion is produced of the Moon's describing an ellipse about the Earth fixed. Also, at the next conjunction the spectator sees the Moon in the line $T'M'$, or, being unconscious of his own motion, in TN parallel to $T'M'$, i. e. between two successive conjunctions the Moon seems to have described more than a complete circle by the $\angle LTN = TST'$, or the synodic period is longer than the sidereal.

376. From the comparison of the Sun's mean horizontal equatoreal parallax, $8'',57116$, with that of the Moon which by the latest determination is $57'.0'',9$, we find that mean dist. of Sun from the Earth $= \frac{3420,9}{8,57116}$ mean dist. of Moon $= 399,1$ × mean dist. of Moon = 400 times Moon's mean dist. nearly.

377. To find the sidereal time occupied by the passage of the Moon's semi-diameter across the meridian.

Let apparent semi-diameter of Moon expressed in time $= h$, then angle which it subtends at the Pole $= \frac{h}{\cos\delta}$; let λ = increase of Moon's R.A. in 1 sidereal second, expressed in time; then $1 - \lambda$ is the hour angle described by the Moon's center in one sidereal second expressed in time. Therefore, if t = number of seconds occupied by the Moon's radius in passing the meridian,

$$t(1-\lambda) = \frac{h}{\cos\delta}.$$

378. The Moon's latitude changes continually during the course of a month. It is sometimes 0, when the Moon must be in the ecliptic; after this it gradually increases, till in about 7 days it attains its maximum $5^{0}.8'.47'',9$, when it again diminishes by the same law by which it increased, and at the end of 14 days becomes 0, when the Moon again crosses the ecliptic; after which the latitude runs through the same course of variation on the south side of the ecliptic. By the comparison of the observed latitudes of the Moon it may be proved, as it was

for the Sun (Art. 168.), that the Moon's orbit lies in a plane cutting the ecliptic in the points where the Moon's latitude is 0.

379. *To find the inclination of the Moon's orbit to the ecliptic, and the place of the line of nodes.*

Let β and β' be two latitudes of the Moon on successive days, before and after passing the node, l and l' the corresponding longitudes on the ecliptic, and n the longitude of the node; then, considering the change of longitude and latitude proportional to the time,

$$\frac{n-l}{l'-l} = \frac{\beta}{\beta+\beta'}, \quad \therefore n = l + \frac{\beta}{\beta+\beta'} \cdot (l'-l).$$

Also if I be the inclination, then, as in Art. 342,

$$\sin(l'-n) = \tan\beta' . \cot I; \ \therefore \cot I = \sin(l'-n)\cot\beta'.$$

The inclination may also be deduced by observing the Moon's latitude on successive days when near its maximum, and determining by interpolation the maximum latitude, which will be the inclination of the Moon's orbit.

380. If the Moon's nodes are determined at different times, they are found not to be fixed; but to have an uniform retrograde motion of $19^0. 20'. 33'',46$ in 365 days, with respect to the stars, and therefore a tropical motion of $19^0. 19'. 43'',23$, in the same time; and besides this mean motion, are subject to inequalities, of which the greatest is

$$(1^0. 30'. 26'') . \sin 2 (\odot - \Omega);$$

so that if Ω be the longitude of the ascending node, determined from its position at a known epoch and the uniform increase of $19^0. 19'. 43''$ in 365 days, the true longitude of the node

$$= \Omega + (1^0. 30'. 26'') . \sin 2 (\odot - \Omega).$$

The inclination has also a small inequality

$$= (8'. 47'') . \cos 2 (\odot - \Omega),$$

to be added to the mean inclination which remains constant, if the secular change in the position of the ecliptic be omitted.

381. *To find the Moon's synodic and sidereal periods.*

The time of the Moon's conjunction with the Sun may be found as that of a planet was (Art. 357); and from the observed interval between two conjunctions, the synodic period may be deduced. This will be variable on account of the inequalities of the Sun and Moon's motions; but if the whole time between two distant conjunctions be divided by the number of revolutions made by the Moon in the interval, the mean synodic time will be determined with great accuracy.

The synodic time thus deduced $= 29^d.\ 12^h.\ 44^m.\ 2^s{,}86$,

and substituting this quantity for S in the equation $p = \frac{P.S}{S+P}$, where P is a sidereal year and both the Sun and Moon are supposed to revolve about the Earth, we obtain the mean sidereal period

$$p = 27^d.\ 7^h.\ 43^m.\ 11^s{,}54.$$

If P be a tropical year, the tropical period of the Moon will be found.

382. As there exist very ancient records of solar eclipses, which can only take place at the time when the Moon is near conjunction, the mean synodic time, and mean motion of the Moon, may be deduced from the comparison of some of these; and it is found in every case, that the more ancient the times of conjunction from which the synodic period is deduced, the greater is the period, so that the Moon's mean motion must have been continually accelerated. Laplace, who first detected the cause and law of this acceleration from the theory of gravity, has proved that the quantity

$$10'',207\,t^2 + 0'',00185\,t^3,$$

must be added to the Moon's mean longitude, to give the true mean longitude for the year $1800 + t$.

383. *To find the elements of the lunar orbit.*

The Moon may be proved in the same manner as the Sun, (Art. 57.) by observations of the apparent diameter to describe an ellipse, but considerably more eccentric; and as the variation of this diameter is much greater than the Sun's, this circumstance may be made use of in ascertaining the elements with some accuracy. The *Place of the Node*, the *Inclination of the Orbit*, and the *Mean Motion*, have already been determined. To find the *Longitude of the Apogee*, let the Moon be observed when its diameter is least, then the longitude of the Moon is the longitude of the apogee; and if the place of the apogee be determined in this manner after several revolutions of the Moon, its motion may be determined, and the place found with more accuracy.

The eccentricity $= \frac{\Delta - \Delta'}{\Delta + \Delta'}$, Δ and Δ' being the Moon's greatest and least apparent semi-diameters (Art. 311.); but as the Moon's apparent diameter can rarely be observed on account of the imperfect illumination of one of the limbs, this method is of little value.

The mean distance may be found from the parallax. If p be the horizontal parallax, when the Moon's apparent diameter is δ, determined as in Art. 257,

$$\text{the horizontal parallax at the least distance} = p \cdot \frac{\Delta}{\delta},$$

$$\text{................................greatest......} = p \cdot \frac{\Delta'}{\delta}.$$

$$\text{Therefore the least distance} = \text{rad. of } \oplus \cdot \frac{\delta}{p\Delta}, \text{ (Art. 247.)}$$

$$\text{...............greatest......} = \text{rad. of } \oplus \cdot \frac{\delta}{p\Delta'};$$

$$\text{therefore the mean distance} = \tfrac{1}{2} \text{ rad. of } \oplus \cdot \frac{\delta}{p} \cdot \left\{\frac{1}{\Delta} + \frac{1}{\Delta'}\right\}.$$

The longitude of the Moon at a given epoch is known from the observed time of the Moon's being in its node

The numerical values of the lunar elements at the beginning of the nineteenth century, were, at midnight, Dec. 31, 1800, Paris time, as follows:

The mean longitude of the Moon ...	$118^{\circ}.\ 17'.\ 8'',3$
.... perigee...	$266^{\circ}.\ 10^{\circ}.\ 7'',5$
.... ascend. ☊	$13^{\circ}.\ 53'.\ 17'',7$
The mean inclination..................	$5^{\circ}.\ 8'.\ 47'',9$
The eccentricity........................	0.0548442
The apparent diameter at mean dist.	0.51861 or $31'.\ 7''$
The greatest apparent diameter	0.55863 or $33'.\ 31'$
The least	0.48942 or $29'.\ 22'$
The mean horizontal parallax	0.95050 or $57'.\ 1'',8$
The sidereal revolution of the perigee	$3232^{d}.\ 57534$
.... node	$6793^{d}.\ 39108$
The mean distance of the Moon......	60.2796. Eq. rad. ⊕
Mass, that of Earth being 1, ...	$1 \div 87.73$
Time of rotation	$27^{d}.\ 7^{h}.\ 43^{m}.\ 12^{s}$
Diameter in miles	2160
The mean sidereal period............	$27^{d},321661$
The mean synodic period............	$29^{d},530589.$

384. The progressive motion of the apogee in longitude found by determining its place at different times, is $40^{\circ}.\ 30'.\ 45'',79$ in 365 days. To the longitude determined from its place at a known epoch, and this mean motion, must be added $(27'.\ 17'')$, × sin Sun's mean anomaly, in order to obtain the true longitude of the apogee.

The motions of the apogee and node, besides their mean values and periodical equations, have also secular equations which are to the secular acceleration of the Moon's mean motion (Art. 382.) as 3 to 1, and .74 to 1, respectively.

By the longitude of the Moon is generally understood the longitude in its orbit, from which the reduced longitude on the ecliptic may be easily found.

385. *The equations of the Moon's motion.*

If the Moon and Earth were the only bodies of the system, the Moon would describe an accurate ellipse about the Earth, and the apogee and nodes would be stationary; but in consequence of the attractions of the other bodies of our system, and of the Sun in particular, this motion is much disturbed, and the longitude of the Moon, determined from the elements upon the hypothesis of an elliptic motion, must receive between 30 and 40 corrections, in order to determine the true place of the Moon with the necessary degree of accuracy. The form of these corrections or *equations* as they are usually called, is determined by Physical Astronomy, and the coefficients are calculated from observation. The three greatest equations, and the first discovered, are

$$\text{the evection} = 1^\circ.\,16'.\,28''.\,\sin 2\left\{(\text{☽} - \odot) - M\right\},$$

$$\text{the variation} = 39'.\,30''.\,\sin 2\,(\text{☽} - \odot),$$

$$\text{the annual equation} = 11'.\,13''.\,\sin m,$$

where ☽ is the Moon's mean longitude, $\odot$ the Sun's true longitude, and m and M the mean anomalies of the Sun and Moon. The evection was discovered by Ptolemy in the first century, and the variation and annual equation by Tycho Brahe.

386. Damoiseau has computed the coefficients for the epoch of January 1, 1801, and has found that the true longitude of the Moon in functions of its mean longitude $nt + \epsilon = \lambda$, is

$$\begin{aligned} v = nt + \epsilon &+ 22639'',7\,.\sin(c\lambda - \varpi) \\ &+ 768'',72 \sin(2c\lambda - 2\varpi) \\ &+ 36'',94 \sin(3c\lambda - 3\varpi) \\ &- 411'',67 \sin(2g\lambda - 2\theta) \\ &+ 39'',51 \sin(c\lambda - 2g\lambda - \varpi + 2\theta) \\ &- 45'',12 \sin(c\lambda + 2g\lambda - \varpi - 2\theta) \\ &+ 2370'',00 \sin(2\lambda - 2m\lambda) \\ &+ 4589'',61 \sin(2\lambda - 2m\lambda - c\lambda + \varpi) \\ &+ 192'',22 \sin(2\lambda - 2m\lambda + c\lambda - \varpi) \\ &- 24'',82 \sin(2\lambda - 2m\lambda + c'm\lambda - \varpi') \\ &+ 165'',56 \sin(2\lambda - 2m\lambda - c'm\lambda - \varpi') \\ &- 673'',70 \sin(c'm\lambda - \varpi') \\ &+ \&c. \end{aligned}$$

The first term of this series is the mean longitude of the Moon, including its secular variation. The second term

$$22639'',7 \,.\, \sin(c\lambda - \varpi)$$

is the equation of the center.

The evection is expressed by

$$4589'',61 \sin(2\lambda - 2m\lambda - c\lambda + \varpi)$$

and is the effect of that part of the Sun's force which acts in the direction of the radius vector.

The term $2370'' \sin 2(\lambda - m\lambda)$ is the variation, and is due to that part of the Sun's force which acts in the direction of the tangent.

The annual equation is expressed by $673'',7 \sin(c'm\lambda - \varpi')$ and is occasioned by a variation in the Sun's distance from the Earth. The causes of the lesser inequalities are not so easily traced.

Time of the Moon's rotation. Librations of the Moon.

387. *The time of the Moon's rotation is equal to the time of its revolution round the Earth.*

The Moon seen through a telescope, appears covered with a great number of spots resembling excavations or prominences, which remain fixed *nearly* in the same position, and have been laid down in maps. These maps serve for all times, and therefore prove that the same face of the Moon is always turned towards the Earth, from whence it follows, that the Moon turns round its own axis from west to east in the time of its motion round the Earth. This will perhaps be made clearer by fig. 19. Plate I, where y is the point on the edge of the visible disk when the Moon's center is at any point c; when the center comes to any other point h, if there was no rotation the point y would be at m (hm being parallel to cy and therefore perpendicular to Tc), but by observation it is at y'; therefore between the positions c and h of her center, the Moon has turned through the angle mhy' = complement of $Thm = hTc$; or the angular velocity about the axis is equal to the angular velocity of the radius vector.

388. It is true that by far the greater portion of the Moon's face turned towards the Earth, remains always the same; but parts near the edges are brought into view and carried out periodically; these effects are called *librations.* In the first place, the angular velocity of rotation is probably uniform, whilst the angular velocity of the radius vector (since the Moon's orbit is an ellipse) is variable; consequently, according as the Moon's motion in her orbit is less or greater than her mean motion, she will have turned on her axis more or less than is necessary to keep the same face directly turned towards the Earth; and therefore, since both her motions of rotation and translation are from west to east, in the former case a portion of the eastern limb, in the latter a portion of the western, will be visible, beyond what was originally turned towards the spectator; this is called the *libration in longitude.* Since the angular velocity about the focus in which the Earth is not, is nearly uniform (Art. 313.), the libration in longitude about that point will be exceedingly small.

Secondly, the Moon's axis of rotation is not exactly perpendicular to the plane of her orbit, but inclined to it at $84^0.51'.11''$; therefore when the Moon's north-pole is turned towards the Earth, the parts adjacent to it are visible, and at the end of 14 days, these parts are withdrawn, and the south-pole being turned towards us, the parts near to it are in turn brought into view; this is called the *libration in latitude.*

Thirdly, when the Moon rises, a portion of the upper or western limb is seen by reason of parallax, which would be invisible from the Earth's center, and which disappears as the Moon ascends; and similarly a portion of the eastern limb comes into sight as she descends, this is the *diurnal libration.*

Lastly, two spectators may be so situated on the Earth's surface, that the planes cutting off the hemispheres of the Moon visible to each may be inclined to one another at twice the Moon's horizontal parallax or about 2°, and the portions of the Moon's surface visible at each station will differ to that extent. From all these causes more than half the Moon's surface is known to us, and maps of it have been constructed, wherein the positions of the principal spots are laid down almost with as much accuracy as many of the most remarkable places on our own globe, being referred to that meridian of the Moon which passes through her poles and the diameter which always points to the Earth.

389. Besides these librations which are only apparent, and do not affect the Moon's uniform rotation, there is also a real motion of the Moon's equator. The inclination of the Moon's equator to the ecliptic, may be determined in the same manner as we shall determine the inclination of the Sun's equator, at the end of this Chapter; and it has been found, that a plane drawn through the Moon's center, parallel to the ecliptic, lies between the plane of the Moon's equator and the plane of the Moon's orbit, and that these three planes always intersect in the same line; therefore, as the line of nodes of the Moon's orbit makes a complete revolution in $6793^d.39$, the intersection of the Moon's equator and the ecliptic must revolve in the same time, and the poles of the Moon's orbit and equator must describe small circles round the pole of the ecliptic in such a manner, that these three poles lie always in the same great circle.

The inclination of the Moon's equator to the ecliptic is $1^0.28'.25''$.

The Moon's Phases.

390. We proceed next to shew that the various appearances which the Moon affords are exactly such as would be presented by an opaque body illuminated by the Sun's rays in the different parts of its orbit.

To find an expression for the illuminated part of the Moon's disk.

Let S, M, E (fig. 60.) be the respective centers of the Sun, Moon and Earth, and Mm perpendicular to SE; then, as in Art. 372, if 2Δ be the Moon's disk, and P the phase, or illuminated part of it,

$$P = \Delta\,.\,\text{ver-sin}\,SMO = \Delta\,\text{ver-sin}\,(S+E).$$

But $\angle S = \dfrac{Mm}{SM}$ nearly, $= \dfrac{EM\,.\,\sin E}{SE} = 60\,.\,\dfrac{\text{rad.}\,\oplus}{SE}\,.\,\sin E,$

$$\text{and } \frac{\text{rad.} \oplus}{SE} = \text{the Sun's horizontal parallax } (8'',57);$$

$$\therefore S = 8',57 \sin E,$$

$$\text{and therefore } P = \Delta \,.\, \text{ver-sin}\,(E + 8',57 \sin E).$$

To determine E, draw Mn perpendicular to the plane of the ecliptic, join En, and let abc be a spherical triangle described about E with radius 1, then $\cos ac = \cos ab \cos bc$ or

$$\cos E = \cos\beta \,.\, \cos\,(☽ - \odot),$$

☽ being the Moon's *reduced* geocentric longitude, and β the geocentric latitude.

391. Hence we may find the phase for different positions of the Moon in her monthly revolution.

In conjunction, $☽ - \odot = 0$, and $E = \beta$;

$$\therefore P = \Delta \,.\, \text{ver-sin}\,(\beta + 8',57 \sin \beta),$$

which is not zero, unless the Moon be in its node.

As E increases, the illuminated part increases, and when

$$E + 8',57 \sin E = 90^0,$$

half the disk is illuminated. E is, in this case, evidently less than 90^0.

$$\text{When } ☽ - \odot = 90^0,\ E = 90^0,$$

and P is greater than Δ, and goes on increasing till opposition, when $☽ - \odot = 180^0$ and $\therefore E = 180^0 - \beta$, and

$$P = \Delta \,.\, \text{ver-sin}\,(180^0 - \beta + 8',57 \sin \beta),$$

which equals the Moon's disk, if the Moon be then in its node.

392. When half the Moon's disk is illuminated, $SME = 90^0$, and the boundary of the illuminated part is a straight line; if, therefore, the angle E be observed when the boundary of light is a straight line,

$$EM = SE \cos E, \text{ or}$$

$$\text{the Sun's parallax} = \text{the Moon's parallax} \times \cos E.$$

This method of determining the Sun's parallax from the Moon's, was employed by the ancients; but it does not admit of accuracy, from the difficulty of observing the exact time when the boundary of light and darkness is a straight line.

The Earth, to an inhabitant of the Moon, must present nearly the same phases as the Moon does to us, and if P' be the visible illumined part of the Earth's disk, and $2\,\Delta'$ the apparent disk seen from the Moon,

$$P' = \Delta' \,.\, \text{ver-sin}\,(180 - E) = \Delta' \,.\, (1 + \cos E),$$

$$\text{and} \quad P = \Delta \,.\, \text{ver-sin}\,(S + E) = \Delta \,.\, (1 - \cos E), \text{ nearly,}$$

since the greatest value of S is $8',57$. Hence when $E = 0$,

$$P = 0, \text{ and } P' = 2\,\Delta'.$$

As E increases, P increases and P' diminishes, and when $E = 180^0$,

$$P = 2\,\Delta, \text{ and } P' = 0.$$

393. In order to make the law of variation of the Moon's phases clearer suppose that body to describe a circle in the plane of the Ecliptic about the Earth (fig. 19, Plate I.); this supposition does not differ so much from the truth as to alter the nature of the appearances presented; the $\angle\, TSc$ will have its greatest value when $TcS = 90^0$, and its sine will then $= \dfrac{Tc}{TS} = \dfrac{1}{400}$, Art. 376, hence it may be neglected, and we may say that the breadth and area of the Moon's phase varies as the versed sine of the angle of elongation STc. As the Sun's distance from the Earth is so great compared with the Moon's distance, the planes separating the light and dark hemispheres of the Moon may be considered parallel to one another in all positions of the Moon; and the part of the Moon turned towards the Earth will always be nearly that within the circle described by her center. Hence if n, c, h, &c. represent the sphere of the Moon in different positions, the shaded part will represent the obscure hemisphere; and in each case, only so much of the enlightened side as falls within the circle nch will be visible at T. When the Moon is at n, or the elongation is 0, the whole of the enlightened side is turned from the Earth, and the phase is N; when she comes to c, that part of the

illumined surface which can be seen from T is the zone xcy, and the phase or appearance presented, as has been explained, will be the crescent C, the line joining the cusps being perpendicular to the plane of the paper; similarly when the Moon is at h, or the elongation is 90°, the phase is the half Moon H; at g the phase is gibbous as G, and when the Moon is in opposition at f, the phase is the full Moon F. Through the remainder of the orbit, as the angle of elongation diminishes, the same phases G', H', C', recur in a reverse order; when for instance the Moon is at c', as much west of the Sun as she was east of him at c, the phase is an equal crescent C' with its convexity turned towards the Sun; hence at c and c' the convexities are turned in different directions with respect to T. The moment after the Sun and Moon cease to be in the same straight line, some portion of the illumined surface of the Moon is turned towards the Earth; we do not however see it immediately, both because the versed-sine increases at first more slowly than the angle itself, and because by reason of the Moon's proximity to the Sun, the light she reflects is not sufficient to be distinguished from that emanating directly from the Sun.

394. Similarly, the Earth's surface will reflect light to the Moon, and will present the same appearances; and the breadth and area of the Earth's phase will vary as the versed-sine of the Earth's elongation ScT. Since the sum of the angles STc, ScT is very nearly 180°, the Moon's and Earth's phases will be supplementary to one another; i. e. the *illumined* portion of the Moon will always bear the same ratio to the whole disk, that the *unillumined* portion of the Earth bears to its whole disk: hence at new Moon the whole of the Earth's disk will be enlightened, and at full Moon the whole will be dark. The effects of *Earthshine* are witnessed at the beginning and end of every month, when the illuminated part of the Moon has the shape of a slender crescent; for then the remaining part of the disk is often rendered faintly visible by it, and, by the usual effects of diminished light, appears of smaller dimensions; this does not happen at other times, both because the light which the Earth furnishes is then less, and because the portion thereof, which the obscure part of the

Moon's disk transmits back to us, cannot be distinguished from the light which the illumined part reflects directly from the Sun.

Jupiter's Satellites.

395. Jupiter, as has been stated, when viewed through a telescope is perceived to be accompanied by four small bodies which never separate far from him, but oscillate to and fro, sometimes passing before him and casting shadows on his disk, sometimes disappearing behind him, and sometimes being eclipsed in his shadow at a distance from him; they are called, in the order of their distances from him, the first, second, third, and fourth Satellite. Their orbits, in which they revolve from west to east, are nearly in the plane of Jupiter's equator, and they consequently always appear nearly in a straight line; this might be expected from considering the immense quantity of prominent matter at Jupiter's equator, the polar being to the equatoreal radius as 1 to 1,0785 or as 13 to 14 nearly. From the same cause probably, the orbits of the two first satellites are circular; the third and fourth, being further removed from its influence, move in orbits with very small eccentricities.

Jupiter throws a conical shadow behind him relatively to the Sun, in which the three first satellites, on account of their orbits being nearly in the plane of his equator are always immersed at their heliocentric conjunctions with him, i. e. in every revolution; but the greater inclination of the orbit of the fourth, together with its greater distance, renders its eclipses not quite so frequent. As the satellites are only luminous by reflecting the Sun's rays, they will disappear when they enter the shadow and reappear on the other side of the shadow, after a certain time. If their orbits were in the plane of Jupiter's orbit, they would pass through the axis of the cone at each eclipse, and the duration of the eclipses would always be the same, supposing the orbits circular; but since Jupiter's distance from the Sun and consequently the dimensions of the conical shadow alter, and since all the orbits are more or less inclined to the plane of his orbit, the duration of the eclipses varies. If the heliocentric conjunction of a satellite happen when it is in the node of its orbit, the eclipse will be central and the duration longest; in other cases the orbit of the satellite will be a chord of the section of the shadow more or less great, and the duration of the eclipse will vary to the same extent, being always less than that of a central eclipse.

396. The angle *SIE* (fig. 63.) contained between the radius vector of Jupiter and his distance from the Earth, or the annual parallax of Jupiter, varies about from 0^0 to 12^0; this is the cause of great variations in the angular distances of the satellites from Jupiter at which the eclipses take place, and in the phenomena they exhibit. The beginning and end of the eclipses of the 3rd and 4th satellite can generally be observed from the Earth, both taking place on the same side of Jupiter. For let *S*, *E*, *I* be the centers of the Sun, Earth, and Jupiter, *mn* a portion of the orbit of the third or fourth satellite falling within the shadow; then the immersion and emersion are seen in the directions *Em*, *En*, both clear of Jupiter's body. The other two satellites are so near to the body of Jupiter that the immersions only can be seen, as at *a*, in the direction *Ea*, when the Earth is at *E* west of the line joining Jupiter and the Sun, or before their geocentric opposition; the emersion, which would be seen in the direction *Eb*, being hid by Jupiter. After opposition, when the Earth is at *A* east of the line joining the centers, the emersion is visible but the immersion concealed. The angular distance *mEI* from Jupiter, at which the satellites disappear, will vary according to the relative positions of the Earth, Sun, and planet; when Jupiter is near opposition, they vanish close to his disk. To have the eclipses visible, the Sun should be at least 8^0 below the horizon, and Jupiter not less than 8^0 above it at the same time.

397. Besides the eclipses, the transits of the satellites over Jupiter's disk are observed; their shadows are projected on the disk in the form of black spots which describe cords of the disk, and of course cause an eclipse of the Sun at Jupiter's surface. Shortly after the satellite passes *a'* its shadow will fall upon Jupiter's disk in the direction *Sa'*; but the satellite itself will not appear to the Earth at *E* to enter on the disk till it comes up to the line drawn from *E* to touch the east

side of the disk; i. e. before Jupiter's geocentric opposition a satellite will follow its shadow over the disk; and similarly, after opposition, a satellite will precede its shadow over the disk. There is reason to think that the satellites always turn the same face to the primary, and consequently that each revolves about its axis in the same time as it completes its circuit about Jupiter; this is inferred from the marked changes in respect of brightness, to which they are subject, and which happen periodically according to their positions relative to the Sun.

398. *To find the period of a satellite round Jupiter.*

Let the beginning and end of an eclipse of the satellite be observed, then the time of the middle of the eclipse may be found, which is nearly the time of the satellite's heliocentric conjunction with the primary. If the time of another conjunction be found in the same manner, the whole time between, divided by the number of conjunctions in the interval, gives the synodic time (S) of the satellite and Sun about Jupiter. From this, the sidereal period round Jupiter $= \dfrac{PS}{S+P}$ is known; P being Jupiter's sidereal period, or the Sun's apparent sidereal period about Jupiter.

399. From the above, the mean motion in longitude (m) of a satellite round Jupiter may be found; and if l be the Jovicentric longitude of a satellite at the middle of an occultation by Jupiter, the longitude at any time t from this $= l + mt$. But at the middle of a satellite's occultation by Jupiter, its jovicentric longitude = the geocentric longitude of Jupiter, which can be determined; whence the jovicentric longitude of a satellite at any time is known. This supposes, indeed, that the motion in longitude of the satellite is uniform, which is very nearly the case, since the orbits are nearly circular.

400. If l, l', l'' be the jovicentric longitudes of the three first satellites at any time, it is found that $l - 3l' + 2l'' = 180^0$ very nearly. From this curious result it follows that the three first satellites cannot be eclipsed at the same time; for

if the three satellites were eclipsed together, we ought to have at this time $l = l' = l''$; but if

$$l = l',\ l'' - l' = 90^0, \text{ and if } l = l'',\ l - l' = 60^0.$$

$$\text{Also since } l - 3l' + 2l'' = 180^0;$$

$$\therefore\ d_t l + 2d_t l'' = 3d_t l',$$

where $d_t l$, $d_t l'$, $d_t l''$ represent the motions of the three satellites; this shews that if the mean angular velocity of the first satellite be added to twice that of the third, the sum will equal three times that of the second.

These equations are so exact, that from the earliest observations they have always been satisfied; and it results from the theory of these bodies, that if they had not been exact in the origin of their motions, the mutual attractions of the bodies would have made them so.

401. If an eclipse of a satellite be observed when Jupiter is nearly in geocentric opposition with the Sun, and therefore nearest the Earth, and from the known synodic motion of the satellite, the following eclipses be calculated, they are observed to happen always later than the calculated time; and the difference goes on increasing, till Jupiter is in geocentric conjunction with the Sun, and therefore furthest from the Earth, when it attains its maximum $16^m . 26^s$. From this Römer conjectured, that the eclipses were not seen by us at the instant of happening, but that light takes $8^m . 13^s$ to pass over a radius of the Earth's orbit. This hypothesis fully explains all the appearances, and also accounts for the change of place in the fixed stars treated of in the Chapter on the aberration of light.

402. *To find the line of nodes of the orbits of Jupiter and one of his satellites.*

When the duration of an eclipse of a satellite is greatest, the satellite must pass through the middle of Jupiter's shadow, which lies in the plane of Jupiter's orbit; therefore, the satellite must be in, or very near its node, at the time of heliocentric conjunction, and the heliocentric longitude of the node must be the same as that of Jupiter; this latter is known from the time of the observation; therefore, the direction of the line of nodes is determined. The method is not applicable to the two first satellites, the duration of whose eclipses cannot be observed; but if the immersion be observed, and the time of the satellite's heliocentric conjunction be taken for the middle of the eclipse, half the time of an eclipse may be determined, and observing when this is greatest, the rest of the operation will be the same as before.

403. *To find the inclination of a satellite's orbit to the plane of the orbit of Jupiter.*

Let DCE (fig. 64.) be a section of Jupiter's shadow at the distance of the satellite, made when the duration of the eclipse is near its minimum, DCN the path of the satellite intersecting Jupiter's orbit AN in N; draw AB perpendicular to DC which may be considered as a straight line, and let t be half the duration of the eclipse, and T of the longest eclipse; then

$$BC = \frac{t}{T}. AE \text{ nearly, and } AB = AE. \sqrt{1 - \frac{t^2}{T^2}}.$$

But, σ being the synodic time of the satellite, $AE = \frac{T}{\sigma} 360^0$;

$$\therefore\ AB = \frac{360^0}{\sigma}\sqrt{T^2 - t^2} = \frac{1}{m}. 360^0 \text{ suppose};$$

and AN = the longitude of the node – the heliocentric longitude of Jupiter $= n - l$; therefore, from the spherical triangle ABN,

$$\sin \frac{1}{m} 360^0 = \sin (n - l) . \sin N;$$

$$\therefore\ \sin N = \frac{\sin \frac{1}{m} 360^0}{\sin (n - l)}.$$

The above method is not quite accurate, since Jupiter is much compressed at its poles, and the minor axis, which is nearly perpendicular to the ecliptic, is to the major as 13 to 14. In consequence of this, the section DCE, seen from the Sun, is not circular, and an inclination more near the truth will be obtained by diminishing the value of N, found above, in the ratio of 13 to 14.

404. The sidereal revolutions of the 1st, 2nd, 3rd and 4th Satellite are found to be, respectively,

$1^d.7691$, $3^d.5511$, $7^d.1545$, $16^d.6887$,

and the mean distances in equatoreal semi-diameters of Jupiter,

6.0485, 9.6235, 15.3502, 26.9983,

between which, the proportionality of the squares of the periods to the cubes of the mean distances may be observed. The mean distances however are not yet well ascertained.

The mass of Jupiter, next after the elements of the planetary orbits, is the most important numerical value for the explanation and prediction of the phenomena of the solar system; according to the latest determination by observing the elongations of the fourth satellite, Jupiter's mass $= \frac{1}{1048.7}$.(Sun's mass.)

Jupiter has also several dark stripes parallel to his equator which are called his Belts; they often change their appearance, and seem to resemble clouds.

The Satellites and Ring of Saturn.

405. Saturn has seven satellites, none of which, except the two exterior ones, can be seen without powerful telescopes; the most distant is by far the largest, being probably not much smaller than Mars, and its orbit is considerably inclined to the plane of the ring, whilst the orbits of the rest nearly coincide with that plane; the two interior ones have never been seen except under peculiar circumstances, and with the aid of the most powerful instruments that have ever been constructed; their distances, expressed in parts of Saturn's equatoreal radius, and their sidereal periods, are

Distances.	Sidereal Period.
3.35	$0^d.943$,
4.30	1.370,
5.28	1.888,
6.82	2.739,
9.52	4.517,
22.08	15.945,
64.36	79.330.

406. Saturn is also accompanied by a broad circular ring of no considerable thickness, inclined at an angle of about $28^0.22'$ to the ecliptic, in the center of which Saturn seems suspended without touching the inner boundary of the ring. The ring is illuminated by the Sun, and is visible to us whenever the Sun and Earth are elevated on the same side of it; and becomes invisible, 1st, when its plane passes through the Sun as only the edge is then illuminated; 2nd, when its plane passes through the Earth when the edge alone is seen; 3rd, when the dark side of the ring, upon which none of the Sun's rays fall, is turned to us. The angular magnitude of the major axis of the ring at the Planet's mean distance is $40'',095$. The ring revolves in its own plane in the short period of $10^h.\,29^m.\,17^s$.

407. Since the plane of the ring is not perpendicular to the line joining the centers of the Earth and Saturn, the two circular boundaries will appear to be two similar ellipses, and the ratio of the axes may be determined in a given position of the ring.

Take Saturn's center (fig. 65.) for the center of the sphere, and let the plane parallel to the ecliptic cut the sphere in the great circle ΥNR, Υ being the first point of Aries, and let the plane of the ring cut the sphere in the arc NA; let T be the place of the Earth seen from Saturn on the sphere, and draw TA, TR, perpendicular to NA, ΥR, respectively. Then TA measures the angle at which the line joining the centers of the Earth and ring is inclined to the plane of the ring, and since the ring will appear projected on a plane perpendicular to this line, $90^0 - TA$ is the inclination of the plane of the ring to the plane of projection; therefore, if $2a$ be the diameter of the ring or major axis of the projection, and $2b$ the minor axis,

$$b = a \,.\, \cos(90^0 - TA) = a \,.\, \sin TA.$$

Let λ be the geocentric longitude of Saturn's center; therefore ΥR, the Saturcentric longitude of the Earth, $= 180^0 + \lambda$, let ΥN, the longitude of the ring's node, $= n$;

$$\therefore\ NR = 180^0 + \lambda - n.$$

Let ANR, the inclination of the ring to the ecliptic, $= \nu$. Then, since the ecliptic has been supposed to pass through

Saturn's center, if β be the geocentric latitude of Saturn, TR will equal $-\beta$. Hence, knowing NR, RT, and the angle ANR, AT may be found, in the same manner as the latitude from the given right ascension, declination, and obliquity. Thus, making $RNT = \phi$, we obtain

$$\frac{b}{a} = \sin AT = \sin NT \sin ANT$$

$$= \sin NT \sin \nu \cos \phi + \sin NT \cos \nu \sin \phi$$

$$= \sin \nu \cot \phi \sin RT + \cos \nu \sin RT, \because \sin \phi \sin NT = \sin RT;$$

but $\sin NR = \cot \phi \tan RT$, or $\sin (n - \lambda) \cos \beta = \cot \phi \sin RT$,

$$\therefore \frac{b}{a} = \sin \nu . \cos \beta . \sin (n - \lambda) - \sin \beta . \cos \nu.$$

If the ratio of the apparent axes be known from observation, the above equation gives the value of ν.

In the Nautical Almanac the quantities necessary for determining the geocentric position, magnitude, and appearance of Saturn's ring are given at intervals of 40 days throughout the year.

408. Six satellites are believed to attend Uranus, of two of which, the 2nd and 4th, the existence is certain. They have the peculiarity that the planes of their orbits are nearly perpendicular to the ecliptic, being inclined at about 78°. 58′ to that plane; and in these orbits their motions are from east to west, or retrograde. With the exception of the two interior satellites of Saturn, the satellites of Uranus are of all celestial objects the most difficult to obtain a sight of. Their distance from Uranus, in parts of the radius of Uranus, and their sidereal periods, are

Distances.	Sidereal Periods.
13.12	$5^{d}.89$,
17.02	8.71,
19.84	10.99,
22.75	13.46,
45.51	38.07,
91.01	107.69.

From the difficulty of observing them, little is known of the other elements of these satellites, or of those of Saturn. They observe like those of Jupiter and Saturn the proportionality of the squares of the periods to the cubes of the mean distances.

Rotation of the Sun and Planets.

409. If a remarkable spot, observed on the disk of any planet or satellite at different times, has changed its apparent place more than naturally arises from the combined motions of the Earth and planet, the planet must have revolved round an axis, and the time of rotation may be determined from this circumstance. The times of rotation of the Sun, Mercury, Venus, Mars, Jupiter, Saturn, have been determined, as well as those of Jupiter's satellites and the seventh satellite of Saturn, and the strongest analogy induces us to believe that the rotations of all the other bodies of the system have not been discovered, only from the difficulty of observing them. All these rotations are, as the other planetary motions, from west to east, and it is remarkable that the time of rotation of any satellite = the time of revolution round its primary.

The Sun's rotation is determined from observations made on dark spots which appear often on the Sun's disk, and vanish after some time. They frequently disappear soon after being formed, and at other times continue long enough to be brought into sight on the west side of the Sun's disk, and carried out on the east several times before they finally vanish. The mean interval between two appearances of the same spot is about 27 days, and as this is the synodic time of the Earth and spot, this gives a period of about $25^d\ 12^h$ for the time of the Sun's rotation.

410. *To find the heliocentric latitude and longitude of a spot on the Sun's disk.*

Let S (fig. 66.) be the Sun's center, E the Earth's, P a spot near the edge of the disk referred to the ecliptic by the perpendicular PN.

Let l = the heliocentric longitude of the Earth,

x = ...spot;

$\therefore\ ESN = x - l.$

Let PSN the heliocentric latitude $= y$,

PEN the geocentric latitude $= \beta$,

SEN the difference of geocentric long. of S and $P = E$,

$\Delta =$ the Sun's apparent semi-diameter.

Then $SP \,.\, \sin y = PN = EP \,.\, \sin\beta = SE \,.\, \sin\beta$, nearly;

$$\therefore\ \sin y = \frac{SE}{SP} \,.\, \sin\beta = \frac{\sin\beta}{\sin\Delta},$$

$$\text{and } \frac{\sin(x-l)}{\sin E} = \frac{NE}{SN} = \frac{\cos\beta}{\cos y} \,.\, \frac{EP}{SP};$$

$$\therefore\ \sin(x-l) = \frac{\sin E \cos\beta}{\sin\Delta \cos y} = \frac{\sin E \,.\, \cos\beta}{\sqrt{\sin^2\Delta - \sin^2\beta}}$$

$$= \frac{\sin E \,.\, \cos\beta}{\sqrt{\sin(\Delta+\beta) \,.\, \sin(\Delta-\beta)}}.$$

411. *To find the inclination of the Sun's equator to the ecliptic, and the time of the Sun's rotation.*

Suppose E (fig. 67.) the pole of the ecliptic, P of the Sun's equator, A, A', A'' heliocentric places of the same spot, observed at three different times; join EA, EA', EA'', PA, PA', PA''; then EA, EA', EA'', the co-latitudes, and AEA', AEA'', $A'EA''$, the differences of longitude may be determined by the last article from geocentric observations; therefore, in each of the triangles AEA', AEA'', $A'EA''$, two sides and the included angle are given, from which the bases, and the angles at the bases, may be determined; therefore, the angles $A'AA''$, $AA'A''$, $AA''A'$, which are the sums of two angles at the bases, may be found. Now, because P is the pole of the Sun's equator, parallel to which the spot revolves,

$$PA = PA' = PA'',$$

and if PR be perpendicular to AA'', $AR = \frac{1}{2} AA''$.

$$\text{Now } \tan AP = \frac{\tan AR}{\cos PAR};$$

let $2S = A + A' + A''$

$$= 2\,PAR + 2\,PA'A + 2\,PA'A'' = 2\,PAR + 2\,A\,;$$

$$\therefore\ PAR = S - A';$$

also $\tan AR = \tan \frac{1}{2} AA'' = \sqrt{\dfrac{-\cos S \cos (S - A')}{\cos (S - A) \cos (S - A'')}}$;

$$\therefore \tan AP = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - A') \cos (S - A'')}},$$

and $\cos PE = \cos AP \cos AE +$

$$\sin AP \sin AE \cos (EAR - PAR),$$

which gives PE, the inclination of the Sun's equator to the ecliptic, about $7^0.30'$. Also

$$\sin \frac{APA''}{2} = \sin APR = \frac{\sin AR}{\sin AP},$$

which gives APA''; if therefore t be the interval between the observations at A, A'', and P the time of rotation,

$$P = \frac{360^0}{APA''} . t,$$

which determines P.

CHAPTER XVI.

ON ECLIPSES, AND OCCULTATIONS OF FIXED STARS BY THE MOON.

Explanation of Lunar Eclipses.

412. We next proceed to the description and explanation of the striking phenomena of the *Eclipses* of the Moon and Sun. It often happens that the full Moon suddenly grows less bright, till a portion, or the whole, of the disk becomes quite obscured, or, if seen at all, is of a faint dusky red; the separating line being successively different parts of a circle, and not a complete semicircle as in the ordinary phases, and the brightness diminishing towards the part which has disappeared. After a short time, the obscured part begins to diminish in magnitude, the whole disk again becomes visible, at first less bright, but soon recovering its former brilliancy. This must be caused by the interposition of some opaque body between the Sun and Moon, and that body is the Earth; for eclipses never happen except at the time of opposition, when the Earth occupies that situation; and the Earth being opaque has of course a conical shadow tapering to that point in the line joining the centers of the Sun and Earth at which the apparent diameters of these bodies are equal, and which is consequently distant from the Earth about $3\frac{1}{2}$ times the Moon's distance; also the breadth of the shadow at the part where the Moon crosses it, is about $\frac{8}{3}$rds of the Moon's diameter; hence if the Moon moved in the ecliptic, the Earth's shadow is long enough, and broad enough at the part where the Moon would cross it, to totally eclipse the Moon at every opposition.

413. Lunar eclipses, however, do not happen every full Moon; and the reason is that the Moon's orbit does not coincide with the Ecliptic. For, since the plane of the Moon's orbit is inclined at about $5^{\circ}.9'$ to the plane of the ecliptic, the Moon in her oppositions is often considerably elevated above, or depressed below the plane of the ecliptic, so as not to be interfered with by the Earth's conical shadow; and in fact can

only be so, when at the same time she is in opposition, she is also very near one of her nodes. When the Moon at opposition is so near the node as to be wholly immersed in the shadow, the eclipse is total, when only a part of the disk is immersed the eclipse is partial; and it is the Moon's proximity to the node at the time of opposition, which occasions all the varieties observed in eclipses.

414. The above will be made clear by fig. 20, Plate I, where *MN* represents a portion of the Moon's orbit, and *Nem* its projection on the plane of the ecliptic, in which plane are situated *N* the node, *S* and *E* the centers of the Sun and Earth, and *SEeC* the axis of the Earth's conical shadow *DCF*. The circles *AGB*, *DHF* are sections through their centers of the Sun and Earth, made by a plane perpendicular to the ecliptic; *AD*, *BF* are common tangents to these circles on the *same* side of the line joining their centers, meeting in *C* a point in that line beyond the Earth, since the Sun is the larger body; then the triangle *DCF* revolving about *EC* generates the conical surface *DCF*, within which no portion whatever of the Sun's light can enter. Also *AF*, *BD* are common tangents on *different* sides of *SE*; and the triangle *DKF* generates a conical surface, the produced portion of which to the right of *E*, includes a space exterior to *DCF*, called the *penumbra*, every point of which can receive rays only from a part of the Sun, and the less, the nearer it is to the axis of the cone. Let a plane through *e*, perpendicular to *SE*, cut these cones in the circles *ab*, *cd*; then the portion of the Moon's orbit *pq* will nearly coincide with the plane of the circles, and in describing it, the Moon will receive only a portion of the Sun's light and her brightness will diminish the further she advances into the penumbra; and if any part of the Moon's disk fall within the circle *ab*, so much of it will be deprived of the whole of the Sun's light.

415. The sum of the greatest apparent radii of the Earth's shadow and Moon is $62'.37''$, the sum of the least $51'.23''$; consequently if during her revolution the Moon's center never comes so near the center of the Earth's shadow, or the point of the ecliptic opposite to the Sun, as $62'.37''$, there can be no eclipse; if her center comes within that distance, there may,

and if within 51′.23″, there must be an eclipse. Hence by calculation it appears that if, at the time of full Moon, the node be further than 11°.25′.40″ from the point of the ecliptic opposite to the Sun, there can be no eclipse; this quantity is called the *lunar ecliptic limit.*

If the Moon's center at the moment of opposition be in the node, it will also be in the axis of the Earth's shadow, and the eclipse is called central, and under these circumstances may continue total for two hours; but it is found that the Moon's face still remains more or less faintly visible; the reason of which is that those of the Sun's rays which graze the Earth's surface are bent inwards by the refraction of the Earth's atmosphere, and so thrown upon the Moon's disk.

Description and explanation of Solar Eclipses.

416. The new Moon produces effects on the Earth similar to those which the Earth produces on the full Moon. For although the Moon is incomparably smaller than the Sun, yet she is so much nearer, that her apparent diameter differs little from that of the Sun, being sometimes greater and sometimes less; hence when she is in conjunction with the Sun, she may prevent the whole, or a part, of the Sun's light from reaching places on the Earth's surface. Suppose the centers of the Sun and Moon and the eye of the spectator to be in the same straight line, then if the relative distances of the Sun and Moon at that instant be such that the Moon's apparent diameter is greater than the Sun's, the eclipse will be total; if less, there will be presented a luminous ring formed by the Sun's disk projecting beyond the edge of the Moon's, or the eclipse will be *annular;* if the Moon's center be not exactly in the line joining the Sun's, and the spectator, a portion only of the Sun's disk will be hidden, and the eclipse will be partial.

417. Moreover the elevation of the Moon above the horizon changes her apparent diameter, (Art. 248.); and her parallax, being considerable, causes different observers to see her in points relative to the fixed stars considerably distant from one another, and may augment or lessen the apparent distance of the centers of the Sun and Moon to such a degree, that an eclipse of the Sun may be presented to one observer,

which has no existence to another. In this lies the main distinction between lunar and solar eclipses; in the former there is an actual deprivation of light, and the appearance is not affected by the Moon being seen in different points with respect to the fixed stars; at all places which then have the Moon above their horizon, the time, duration, and extent of the eclipse are the same; in the latter, the intersection of the Moon's conical shadow with the Earth's surface produces a circular black spot which sweeps over the illumined part of the Earth's surface from west to east, and resembles in its effects a cloud carried by a west wind, which hides the Sun only from places nearly below it.

418. Hence it appears that the various distances of the Sun and Moon from the Earth's center, the different positions of the Moon with respect to the node at the time of conjunction, and the different situations of the spectator, will produce great varieties in the observed time, duration and magnitude of a solar eclipse; and the precision with which these particulars can be calculated and predicted, affords a convincing proof of the accuracy of the theory, and the exactness of the tables, depending, as they do, upon the relative situations of the Sun, Moon, and Earth, their velocities, volumes, and parallaxes. A total eclipse of the Sun is of all the celestial phenomena the most imposing; the brightness of day is in a few minutes changed into gloomy twilight, the stars become visible, and all that is seen either of the Sun or Moon is a faint halo surrounding the Moon's invisible disk, supposed to be caused by the Sun's atmosphere; in about five minutes, the Sun emerges with a degree of splendour not equalled even at his rising. The greatest possible duration of a total eclipse of the Sun is $7^m\ 58^s$; and of an annular eclipse $12^m\ 24^s$.

419. The circumstances peculiar to solar eclipses will be better understood by referring to fig. 21, where m is the center of the Moon at the instant of conjunction, and AB, ab, CT, sections through their centers, of the Sun, Moon, and Earth, made by a plane perpendicular to the ecliptic, and aCb a section through its axis of the Moon's conical shadow made by the same plane; then C, the vertex of the shadow, is the point at which the apparent diameters of the Sun and

Moon are equal, and its distance from the Moon Cm will depend upon Sm which has its greatest, and least value, when the Sun is in apogee and the Moon in perigee, and vice versâ; and these values of Cm are respectively 59,73 and 57,76 times the Earth's radius; but in the former case the Moon's distance from the Earth is only 55,96 radii, and in the latter 63,87 radii. Hence when the Moon at conjunction is sufficiently near the node for the axis of her shadow to meet the Earth, there will be a total eclipse (provided her place in her orbit be near perigee,) at all places contiguous to that point of concurrence, within a circle whose radius is something less than $\frac{1}{12}$th of the Moon's radius; such being the dimensions of the Moon's shadow at the Earth's surface under the aforesaid circumstances. If her place be near apogee, the vertex of her shadow will not reach the Earth, but at places contiguous to the point where the prolongation of the axis of her shadow meets the Earth, an annular eclipse will be witnessed. Also if Ab be drawn touching the Sun and Moon on different sides of the line joining their centers, and meeting the Earth in g; then at g the Sun is seen quite clear of the Moon, but at all places between g and h a portion of the Sun's disk is hidden.

420. Again, let M (fig. 58, Plate III.) be the center of the Moon at any instant during a solar eclipse, moving with the excess of her horary motion in longitude above that of the Sun, in the direction BM from west to east; then we may regard ⊙ the center of the Sun as fixed; also let T be the center, supposed fixed, of the Earth revolving about its axis PA in the direction HnH', and let the umbra and penumbra meet the Earth's surface in the curves ab, ns, respectively. Then m, where the axis of the shadow meets the Earth, has a central eclipse; all places within ab have a total eclipse; and all between the curves ab and ns a partial eclipse, varying in magnitude from an internal to an external contact; and those places which are in opposite directions from m see the eclipse upon opposite limbs of the Sun. At places east of the meridian through e the eclipse is ended; because, although the tendency of the Moon's motion is to carry the shadow over them, the tendency of the Earth's rotation, which is far more rapid, is to carry them beyond the shadow; at places west of

the meridian through *w*, the eclipse has not begun. If *M* were immoveable, by the effect of the Earth's rotation, all places between parallels through *a* and *b*, would be brought into the shadow and made to suffer a total eclipse; and all places between parallels through *n* and *a* would see the Sun's southern limb eclipsed; whilst places between parallels through *b* and *s* would see the Sun's northern limb eclipsed; and these appearances would recur every 24^h. But the Moon's motion alters all this. In the first place, the motion in latitude makes the lines of equal phases, such as $R'mR$, which is the line of the central eclipse, and $H'nH$, $K'sK$, which are the northern and southern lines of simple contact of the limbs, to become irregular curves instead of parallels of latitude. And in the second place, the motion in longitude soon puts an end to the eclipse by causing the penumbral cone to turn about ☉ in direction *BM* till it becomes quite disengaged from the Earth's surface; in doing this all its generating lines, beginning with *Ee*, successively become tangents to the Earth's surface in quitting it, and so trace out by their points of contact the irregular oval *HdKl* which is the locus of places that see the eclipse end at sunset.

421. For the better understanding of this, suppose, in the time the penumbral cone takes to revolve from its present position till it begins to quit the Earth, that body to have revolved so much that *d* is the point which *Ee* touches in quitting the Earth; then the tangent plane at *d* (i. e. the horizon at *d*) will touch the Moon's eastern and the Sun's western limb, or the eclipse will end at sunset; and *d* will be the last place that sees the whole of the eclipse in its greatest extent. Afterwards by the continued motion of the Moon, every pair of corresponding generating lines of the penumbral cone on opposite sides of *Ee* will successively become tangents to the Earth's surface, as at *r*, *t*, and the points of contact will trace out all the places *HdK* relative to which the eclipse ends at sunset. When the whole of the eastern boundary of the penumbral cone is clear of the Earth, *Nn*, *Ss* will become tangents at *H* and *K*, and the two extreme westerly places will be determined which witness nothing of the eclipse except a simple contact at sunset on the Sun's south and north

limb respectively. After that, pairs of generating lines of the western side of the penumbral cone on opposite sides of *Ww* will successively become tangents, as at *p* and *q*, and the points of contact will trace out the curve *HpqK* passing through all the places at which nothing more of the eclipse is seen than a simple contact on the Sun's western limb at sunset. And the extreme western point at which even that is seen is *l*, at which the penumbra finally quits the Earth, and the general eclipse ends. When the axis of the shadow becomes a tangent to the Earth, which will be at some point *R*, the last place which sees a central eclipse, and which it does at sunset, is determined; and the line *HRK* is the locus of the places that see the middle of the eclipse at setting.

422. Similarly, at the commencement of the eclipse, when the penumbral cone and also the portion of the Earth's surface $H'K'$ are on the west side of the meridian *ZP*, as the generating lines of the penumbral cone successively come in contact with the Earth's surface, an irregular oval $H'd'K'l'$ will be traced out, being the locus of places that see the eclipse begin at sunrise. The eastern side of the penumbral cone will first come in contact with the Earth at a single point l' and there the general eclipse will begin, but at that place it will be confined to a simple contact at sunrise; and the successive contacts of the generating lines of the east side of the penumbral cone will determine a series of places $H'l'K'$ at which nothing more is seen of the eclipse than a simple contact at rising, for all these places, by the Earth's rotation, will be immediately carried beyond the penumbra. Afterwards the western generating lines will become tangents and determine a series of places $H'd'K'$ at which a simple contact at sunrise will be witnessed; and this will be the beginning of the eclipse at those places, as by the Earth's rotation they will be afterwards carried into the penumbra, and experience partial or total eclipses. And when the penumbral cone has just wholly entered upon the Earth, its western side will touch the Earth in a single point d' which is the first place that sees the eclipse in its whole extent. Also when the axis of the shadow moves up to touch the Earth's surface, as at R', the first place which sees a central eclipse, and which it does

at sunrise, is determined, and $H'R'K'$ is the locus of places that see the middle of the eclipse at rising. By performing the requisite calculations a map of any solar eclipse may be made, in which all that portion of the globe $l'H'nHlKsK'$ where it is visible, and the terrestrial lines of the various phases, and the rising and setting ovals, may be laid down according to the latitudes and longitudes of the places through which they pass.

423. As before, the cause of solar eclipses not happening every new Moon, exists in the Moon's orbit not coinciding with the ecliptic; and the *solar* ecliptic limits are determined in the same manner as those of the Moon; from which it results that if the Moon's distance from the node, when in conjunction, is less than $13^0.42'$, a solar eclipse is certain to happen, but if it exceed 19^0, an eclipse is impossible.

Eclipses will in general be separated by six months, but will occur at all seasons of the year, because the nodes are in perpetual motion along the ecliptic. The synodic period of the Sun and Moon is $29^d,53058$, that of the Sun and the node of the Moon's orbit $346^d,6196$, which are nearly in the ratio of 19 : 223; hence after 223 lunations or 19 revolutions of the node, the Sun and Moon return nearly to the same positions relative to the Moon's nodes, and eclipses will recur in the same order. If, therefore, all the eclipses (usually 29 lunar and 41 solar) which happen in the above period, which is 18^y and 10 or 11 days, be observed and registered in order, we should be able to predict them for the next period of 18 years; but as 223 lunations differ from 19 revolutions of the node by about $0^d,46$, and the Sun and Moon do not move uniformly, this method can only be regarded as an approximation which may be used as a guide to more exact investigations. The cause, possibility, and phenomena of eclipses having now been explained, we proceed to reduce the subject to calculation.

Calculation of Lunar Eclipses.

424. *To find the length of the Earth's shadow, and the apparent semi-diameters of sections of the umbra and penumbra at the distance of the Moon.*

Let ABC, abc (fig. 68.) be sections of the Sun and Earth, made by a plane passing through their centers; BD, CD common tangents meeting in D; BOc, COb common tangents meeting in O; then if the figure revolve round SD, BDC will generate the conical shadow of the Earth into which none of the Sun's rays enter, and mbp, ncq the penumbra which admits only part of the Sun's rays. Let $pmnq$ be a section of these figures, made by a plane perpendicular to SO at the distance of the Moon, and let p be the Moon's horizontal parallax, π the Sun's, μ and σ their apparent semi-diameters, and R the radius of the Earth; then

$$EDB = SEB - EBD = \sigma - \pi;$$

$\therefore$ ED the length of the shadow $= \dfrac{Eb}{\sin EDb} = \dfrac{R}{\sin(\sigma - \pi)}$

The apparent radius of a section of the shadow at the distance of the Moon

$$= MEm = EmB - EDm = p + \pi - \sigma.$$

The apparent radius of a section of the penumbra

$$= MEp = EpO + EOp = p + \pi + \sigma,$$

$$\text{because } EOp = SOB = OBE + OEB = \pi + \sigma.$$

The numerical values of ED, MEm, MEp are subject to changes, from variations in the distances of the Sun and Moon from the Earth; and it is found that

ED varies from 212 to 220 radii of the Earth

MEm .... $37'.\ 42''$ ..... $45'.\ 52''$,

MEp .... $1^{0}.\ 8'.\ 12''$..... $1^{0}.\ 18'.\ 27''$.

Moreover it is found that the Earth's atmosphere deflects and absorbs those rays of the Sun, which pass near the Earth's surface, and thus causes the shadow of the Earth to appear somewhat larger than it would otherwise be. In order to take this effect into account, astronomers consider the radius of the shadow to be one-sixtieth greater than the value just found, or add as many seconds as there are minutes in the radius of the section. As the radius of the shadow is so much larger than the Moon's apparent radius, lunar eclipses can never be annular.

425. It was remarked, that no eclipse can take place when the distance of the center of the Earth's shadow, that is, the point of the ecliptic opposite to the Sun, from the node of the lunar orbit at the time of the Moon's being in syzygy, exceeds a certain quantity called, in lunar eclipses, the *lunar ecliptic limit*, and in solar eclipses the *solar ecliptic limit.* It is of great importance to know its value, because then the mere inspection of the tables will inform us whether, at any syzygy of the Moon, an eclipse is possible or not.

To find the lunar ecliptic limits.

Let ON, LN, (fig. 69.) be portions of the ecliptic and of the Moon's orbit, both considered as straight lines and as described uniformly by the centers of the Earth's shadow and Moon; N the descending node, O the center of the shadow when the Moon is in opposition at L, S the center of the shadow when the Moon is in the node; therefore in the time of moving from L to N, the Moon approaches the center of the shadow by the difference of latitude LO, and recedes from it by the difference of longitude SN. Take SM parallel and equal to LO, and join MN; then if a body describe MN uniformly in the same time as LN and OS are described by the centers of the Moon and shadow, it will approach and recede from the fixed point S in the same manner as the Moon in its orbit does, from the center of the shadow in motion, and its distance from S at any intermediate instant will be equal to the corresponding distance of the centers of the Moon and shadow*. If then we consider the center of the shadow fixed in S the point it occupies when the Moon is in the node, and substitute MN, which is called the *relative* orbit, for the real orbit LN, the distance of the centers of the Moon and shadow, on which the circumstances of the eclipse entirely depend, will undergo precisely the same variations as it does when both are in motion. Hence if Sm be drawn perpendicular to MN, Sm is the nearest approach of the centers of the Moon and shadow;

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* Let M', L', O', be contemporary positions, then $L'M'$ is parallel to LM since it divides MN, LN in the same ratio; also,

$$\frac{L'M'}{LM}=\frac{NL'}{NL}=\frac{\text{time through } O'S}{\text{time through } OS}=\frac{O'S}{OS}, \text{ but } OS=LM, \therefore O'S=L'M';$$

hence the figure $L'S$ is a parallelogram, and $SM'=L'O'$

and taking Sm (Δ) = the greatest semi-diameter of the Moon $15'.45''$ + the greatest semi-diameter of the shadow at the distance of the Moon $45'.52''$, the greatest value of SN or ON, at which an eclipse is possible, will be obtained. To find SN; let m be the Moon's horary motion in longitude, s the Sun's, which is the same as that of the shadow, $f = m - s$, the *relative* horary motion of the Moon and shadow, g the Moon's horary motion in latitude. Then, because SM, SN are described in the same time with the uniform motions g and f,

$$\tan N = \frac{g}{f}; \quad \therefore SN = \frac{\Delta}{\sin N} = \Delta \frac{\sqrt{f^2 + g^2}}{g}.$$

$$\text{Also, } \frac{ON}{SN} = \frac{m}{f}; \quad \therefore ON = \Delta \cdot \frac{m}{f} \cdot \frac{\sqrt{f^2 + g^2}}{g}.$$

The numerical value of ON is $12^0.\ 36'$; if, therefore, the difference of longitudes of the Moon and node at the time of opposition be greater than $12^0.\ 36'$, no lunar eclipse can take place, if $ON = 12^0.\ 36'$ the Moon's disk may just touch the shadow, and if ON is less than $12^0.\ 36'$, there may be a lunar eclipse.

Similarly, if Δ' be taken equal the least semi-diameter of the Moon $14'.40''$ + the least semi-diameter of the shadow $37'.42''$,

$$ON = \Delta' \cdot \frac{m}{f} \cdot \frac{\sqrt{f^2 + g^2}}{g} = 9^0,$$

the least ecliptic limit, is obtained; if, therefore, the Moon at the time of opposition be within 9^0 of the node, there must be a lunar eclipse. Between the distance 9^0 and $12^0.\ 36'$, an eclipse is doubtful, and the exact values of the radii of the Moon and shadow must be found in order to ascertain whether there will be a lunar eclipse or not.

426. *To find the time, magnitude, and duration of a lunar eclipse.*

Let ABB' (fig. 70.) be a section of the Earth's shadow at the distance of the Moon, Sn the path described by its center S on the ecliptic; Mn the relative orbit of the Moon, Mn, Sn being considered straight lines, as in the preceding Art. Draw SO perpendicular to Sn, and Sm to Mn, then O and m

are the places, with respect to S, of the Moon's center in opposition, and at the middle of the eclipse.

Let $a = SB = p + \pi - \sigma$ the radius of the section of the shadow,

$\lambda = SO$, the Moon's latitude in opposition,

$f =$ the relative horary motion in longitude of the Sun and Moon, or the motion in longitude of the Moon in the relative orbit Mn,

$h =$ the Moon's horary motion in the relative orbit,

$g =$ .... in latitude,

$\mu =$ .... semi-diameter;

$$\therefore \tan n = \frac{g}{f}, \text{ and } g = h \sin n.$$

Let M and M' be the places of the Moon's center at the time of the first and last contact; therefore

$$SM = SM' = a + \mu.$$

Now $Sm = \lambda \cos n$;

$$\therefore Mm = \sqrt{(a + \mu)^2 - \lambda^2 \cos^2 n} = M'm,$$

and $mO = \lambda \sin n$.

If, therefore, t and t' be the times from opposition of the first and last contact,

$$t = \frac{Mm - Om}{h} = \left\{\sqrt{(a + \mu)^2 - \lambda^2 \cos^2 n} - \lambda \sin n\right\} . \frac{\sin n}{g},$$

$$t' = \frac{M'm + Om}{h} = \left\{\sqrt{(a + \mu)^2 - \lambda^2 \cos^2 n} + \lambda \sin n\right\} . \frac{\sin n}{g};$$

$$\therefore \text{the } \textit{duration} = 2\sqrt{(a + \mu)^2 - \lambda^2 \cos^2 n} . \frac{\sin n}{g}.$$

The time from opposition, of the middle of the eclipse

$$= \frac{Om}{h} = \frac{\lambda \sin^2 n}{g}.$$

The *magnitude* of the eclipse, or the part of the Moon's diameter immersed

$$= Su - Sv = Su - Sm + mv = a - \lambda \cos n + \mu.$$

The Moon's diameter 2μ is generally divided into 12 equal parts called digits;

$$\therefore \text{ the number of digits eclipsed} = \frac{6}{\mu} \cdot (a - \lambda \cos n + \mu).$$

If $\lambda \cos n$ be greater than $a + \mu$, t and t' are impossible, and no eclipse can take place, as is also evident from the figure.

427. In exactly the same manner it may be proved, if t and t' be the times from opposition, of the centers of the shadow and Moon being at any given distance c,

$$t = (\sqrt{c^2 - \lambda^2 \cos^2 n} - \lambda \sin n) \cdot \frac{\sin n}{g},$$

$$t' = (\sqrt{c^2 - \lambda^2 \cos^2 n} + \lambda \sin n) \frac{\sin n}{g};$$

Hence if $c = p + \pi + \sigma + \mu =$ the radius of the penumbra + the radius of the Moon, the times of the Moon entering and emerging from the penumbra are obtained.

The horary motion of the Moon is about $32\frac{1}{2}'$, and that of the Sun $2\frac{1}{2}'$; therefore the relative horary motion of the Moon is $30'$, and as the greatest diameter of the shadow at the distance of the Moon is $1^0. 31'. 44''$, a lunar eclipse may last more than 3 hours.

428. *To find at what place on the Earth's surface a given lunar eclipse is visible.*

The question is simply to find what places have the Moon above their horizon at a given time, when, from the known position she occupies, part of her surface is deprived of the Sun's light.

Suppose t the Greenwich time of the beginning of the eclipse, or the Sun's hour angle west of the meridian $= 15^0.t$; therefore the Moon's hour angle east of the meridian of Greenwich

$$= (12 - t)\, 15^0,$$

since she is in opposition; let her declination $= \delta$; then a place whose longitude east of Greenwich is $(12 - t)\, 15^0$, and latitude

δ, has the Moon in its zenith at the beginning of the eclipse. If, therefore, about this point as a pole a great circle be described on a common terrestrial globe, all places included between this circle and its pole, will see the beginning of the eclipse. In the same manner from the time of the end, and the Moon's declination at the end of the eclipse, the circle bounding all places which see the end of the eclipse may be found; and those parts of the Earth which are common to the two divisions, have the Moon above the horizon during the whole time of the eclipse.

Calculation of Solar Eclipses.

429. A solar eclipse is the same in its physical circumstances as an eclipse of the Moon, and the preceding formulæ may be changed, so as to apply to it. The objects of calculation are however different in lunar and solar eclipses, being, in the former, to determine merely whether any part of the Sun's light is intercepted, and in the latter, not only this, but also from what part of the Earth it is intercepted; on this account eclipses of the Sun are treated somewhat differently from those of the Moon.

430. *To find the angular distance of the centers of the Moon and Sun at the beginning of a solar eclipse.*

Let RB, rb, MO (fig. 71.) be sections of the Sun, Earth and Moon, made by a plane passing through their centers, and Rr, Bb common tangents to the circles RB, rb. Then when the Moon enters any part of the luminous cone formed round the Sun and Earth, as at O, it intercepts a portion of the Sun's light from reaching the Earth, and the solar eclipse begins; therefore the angular distance of S and M, at the beginning of the eclipse,

$$\begin{aligned} &= SEM = SER + OER + OEM \\ &\qquad = SER + EOr - ERr + OEM \\ &\qquad = \sigma + \mu + p - \pi; \end{aligned}$$

also SEO the radius of the luminous cone at the distance of the Moon

$$= SEM - \mu = \sigma + p - \pi.$$

431. From the figure, it is evident that at *r* where the eclipse begins, the apparent contact takes place in the horizon, the disk of the Moon being wholly above and that of the Sun below it, and that the point of contact is in the same vertical with the two centers. Similarly, the place which first has a central eclipse is that where the line joining the centers of the Sun and Moon first comes in contact with the Earth, and the centers of both bodies are in the horizon; and the same circumstances will occur when the phenomena finally quit the Earth. Also the eclipse begins at *r* some time before it is visible at the center of the Earth, on account of the spectator at *r* being so much raised above *E*, or which is the same thing, on account of the Moon being so much depressed by parallax to a spectator at *r*. For the same reason, when the Moon appears emerging to a spectator at *E*, it may be so depressed by parallax as to have emerged some time at *r*, whilst from the same cause, to a spectator at *b*, the emersion may not have commenced. The Sun is affected by parallax as well as the Moon, although in a much less degree, on account of its greater distance from the Earth; therefore the centers of the Sun and Moon are made to approach or separate by the difference of the lunar and solar parallax.

432. The solar ecliptic limits may be found in the same manner as those of the Moon were, by making *Sm* (fig. 69.) = the greatest semi-diameter of the section of the luminous cone at the Moon + the greatest semi-diameter of the Moon, for the greatest limit; or, = the least semi-diameter of the section + the least radius of the Moon, for the least limit. Hence it appears, that if the distance of the Moon's center, at the time of new Moon, from the true place of the node is

less than $13^0. 42'$, the eclipse is certain,

between $13^0. 42'$ and $18^0. 36'$......... possible,

greater than $18^0. 36'$......... impossible.

For the doubtful case, an eclipse will happen whenever the Moon's latitude when in conjunction $< \sigma + \mu + p - \pi + 25''$.

433. *To find the time and duration of a solar eclipse.*

Adopting the same figure and notation as in the lunar eclipse with the exception of making SB (a) the radius of a section of the luminous cone at the Moon $= \sigma + p - \pi$, we get

The time from conjunction of the first contact

$$= \{\sqrt{(a+\mu)^2 - \lambda^2 \cos^2 n} - \lambda \sin n\} \, \frac{\sin n}{g}.$$

The time from conjunction of the last contact

$$= \{\sqrt{(a+\mu)^2 - \lambda^2 \cos^2 n} + \lambda \sin n\} \, . \frac{\sin n}{g}.$$

$$\text{The duration} = 2 \, \frac{\sin n}{g} \sqrt{(a+\mu)^2 - \lambda^2 \cos^2 n}.$$

As the time of conjunction is known from the tables, the times of the beginning and end of the whole eclipse are thus determined. The duration thus determined is not the duration of the eclipse seen from the center of the Earth, or from any given point of the surface, but the whole duration of the general eclipse, from the time when the Moon begins, to the time when the Moon ceases to intercept any part of the Sun's light, from any point of the Earth's surface.

434. *To calculate the circumstances of a solar eclipse at a given place.*

We must first find the apparent difference at the given place, of longitude and latitude of the centers of the Sun and Moon at any assigned time during the eclipse.

Let Z (fig. 71. bis) be the zenith of the given place, K the pole of the ecliptic $N\Upsilon$, M the Moon near its node, S the true place of the Sun, S' his apparent place depressed in the circle ZS; π his horizontal parallax. Join KS' meeting the ecliptic in s,

$$\text{then } Ss = SS' \cos ZSN = \pi \sin ZS \sin ZSK = \pi \sin ZK \sin ZKS,$$

$$S's = SS' \sin ZSN = \pi \sin ZS \sin ZSN = \pi \sin ZN.$$

Now $ZK = n$, and $♈ N = N$, the altitude and longitude of the nonagesimal are known at any assigned time; let $L' = ♈ S$ be the corresponding longitude of the Sun,

$\therefore$ the Sun's apparent longitude $= L' - \pi \sin n \sin (N - L')$,

and ....latitude $= - \pi \cos n$.

Similarly, because the Moon's latitude during a solar eclipse is very small and the longitude nearly the same as the Sun's, if L, Λ, p, be her true longitude, latitude, and horizontal parallax,

the Moon's apparent longitude $= L - p \sin n \sin (N - L')$

.... latitude $= \Lambda - p \cos n$;

therefore the apparent difference of longitude

$$= L - L' - (p - \pi) \sin n \,.\, \sin (N - L'),$$

and apparent difference of latitude

$$= \Lambda - (p - \pi) \cos n.$$

Now having found, as in Art. 433, the absolute times T and T' of the beginning and end of the whole eclipse; the *true* longitudes of the Sun and Moon, and the true latitude of the Moon corresponding to those times, may be taken out of the Solar and Lunar Tables; and correcting the values so taken for parallax as above, the apparent differences of longitude and latitude of the Sun and Moon may be found. Let l, l' be these apparent differences of longitude, and λ and λ' of latitude for the times T and T'; then if C be the time of apparent conjunction at the place in question, considering the approach of the bodies to one another in longitude uniform during the time $T' - T$, we have

$$\frac{C - T}{T' - T} = \frac{l}{l - l'};$$

$$\therefore\ C = T + \frac{l}{l - l'} \,.\, (T' - T).$$

Also $\frac{l - l'}{T' - T}$ and $\frac{\lambda' - \lambda}{T' - T}$ are the apparent horary approaches of the Sun and Moon in longitude and latitude. Therefore, using these values instead of f and g, and the apparent difference of latitude for the time C instead of λ, all the circumstances of a solar eclipse at a given place may be calculated in the same manner as the lunar eclipse, Art. 426.

435. The circumstances of a solar eclipse have been calculated in the preceding Articles, upon the supposition of the Sun and Moon moving uniformly in right lines during the whole time of the eclipse. This hypothesis was sufficiently accurate for the calculation of the circumstances of a lunar eclipse, in which the times of the beginning and end cannot be exactly observed on account of the penumbra; but in a solar eclipse, where the times of first and last contact can be observed with considerable accuracy, a more exact computation becomes necessary. The time when a solar eclipse commences at a given place, may be found with great accuracy in the following manner.

436. *From the approximate value, to deduce a more correct value of the time when a solar eclipse begins at a given place.*

Let T be the approximate time of the beginning of the eclipse at any place found by Art. 434, $T+t$ the true time; for the time T, let the apparent difference of longitude of the Sun and Moon (l), and the apparent difference of latitude (λ) be found (Art. 434.), and also f and g, the horary increments of these quantities, which may be deduced by subtracting the values of l and λ, calculated for an instant half an hour before T, from those calculated for an instant half an hour after; therefore $l+ft$, $\lambda+gt$ are the values of l and λ at the beginning of the eclipse. Let S (fig. 72.) be the center of the Sun, M of the Moon at the beginning of the eclipse, and P the pole of the ecliptic SN; then

$$SN = l + ft, \quad MN = \lambda + gt;$$

and if μ and σ are the apparent semi-diameters of the Sun and Moon

$$SM = \sigma + \mu;$$

therefore by the right-angled triangle SMN, which may be considered rectilinear, since SN, MN, SM are very small,

$$(\sigma + \mu)^2 = (l + ft)^2 + (\lambda + gt)^2 \ldots\ldots\ldots (1),$$

from which quadratic, t may be found.

The least value of t must be taken, since, by hypothesis, t is very small; the other value belongs to the time of last contact, which may be determined from it, but not so accurately as if the values of l, λ, f, g are calculated from the approximate time of the last contact, and the true time found by an equation similar to (1).

437. If t be very small, or the approximate value T be very nearly the precise time of contact, as generally happens, the square of t may be neglected, and t may be determined from the simple equation

$$(\sigma + \mu)^2 = l^2 + \lambda^2 + 2\,(lf + \lambda g)\,t;$$

$$\therefore\ t = \frac{(\sigma + \mu)^2 - l^2 - \lambda^2}{2\,(lf + \lambda g)}.$$

If μ be affected with a negative sign, the same equation determines the time of internal contact, when the Moon's disk has just entered within that of the Sun.

438. If the Sun and Moon be referred to the equator, and if δ be the Sun's declination at the approximate time of contact T, δ' and a' the differences of apparent declination and apparent right ascension of the centers of the Sun and Moon for the same time, (which may be deduced by correcting the differences of the true declinations and right ascensions for parallax), and f and g the horary increments of δ' and a'; then if M (fig. 72.) be the center of the Moon, S of the Sun, and P the pole of the *equator* at the true time of contact $T + t$,

$$SM = \mu + \sigma,\ MN = \delta' + gt,\ SN = (a' + ft)\,.\cos\delta;$$

$$\therefore\ (\mu + \sigma)^2 = (\delta' + gt)^2 + (a' + ft)^2\,.\cos^2\delta.$$

From which equation, deducing the two values of t, t and t', $T + t$ and $T + t'$ the times of the first and last contact, are obtained as before.

439. *To construct for the places on the Earth's surface where a given solar eclipse is visible.*

The circumstances of an eclipse, which depend only on the distance of the centers of the Sun and Moon, will not be altered if the Sun be considered unaffected by parallax, and the Moon depressed to or from it, by the whole difference of solar and lunar parallax. Let S and M (fig. 73.) be the true centers of the Sun and Moon referred to the celestial vault, as seen from the center of the Earth at any given time during the eclipse. With S as pole, and angular radius $Sv = \mu + \sigma - n$ (n expressed in digits of the diameter of the Sun's disk), describe a small circle round S, take Mv any arc of a great circle less than $p - \pi$, and produce the arc vM to a point Z, such that

$$Mv = (p - \pi) \cdot \sin ZM.$$

Then, at a place the zenith of which is Z, M will be depressed by parallax through Mv, and its apparent distance from S will be

$$Sv = \mu + \sigma - n;$$

or n digits of the Sun will appear eclipsed. In the same way it may be proved that n digits will appear eclipsed at a place the zenith of which is Z', $Z'v'$ being situated symmetrically to Zv on the other side of SM. By taking other points in the arc vv', other places may be determined, in which n digits appear eclipsed, and if T, T' be the two extreme points, determined by taking

$$Mu = Mu' = p - \pi, \text{ and } TM = T'M = 90^0,$$

n digits will appear eclipsed at the same given instant, at all places whose zeniths are situated throughout the whole curve TZT'. If $n = 0$, the locus of the zeniths of places at which a contact just appears is determined; if $n = \mu + \sigma$, $Sv = Su = 0$, and the eclipse is central; in this case, however, only one point is determined, since u and u' coincide. In the same manner, by giving different values to n, all the places which see the eclipse in its various degrees at the instant when the Sun and Moon have the true distance SM, may be found; and by giving SM the values when the Moon begins and ceases to intercept any portion of light, and all intermediate values, and repeating the same process, all places at which the eclipse is visible may be determined.

When the eclipse is first visible,

$$SM = \mu + \sigma + p - \pi;$$

in this case also u and u' coincide, and one place only, 90^0 from M in SM produced, is determined, at which the eclipse is seen when the Sun and Moon are in the horizon.

By determining in this manner a few of the extreme places at which a given eclipse is seen, the intermediate places at which the eclipse is visible, may be easily found on a common globe.

440. *To calculate from the preceding construction the places where a given solar eclipse is visible.*

Through the pole of the equator P (fig. 73.) draw the declination circles PZ, PM, PG, PS, of which PG passes through the zenith of Greenwich; therefore ZPG is the terrestrial longitude of the place whose zenith is Z. Let $MS=c$, $SMv=\phi$, $SMP=\theta$, $Mv=P$, $\tau=$ the time at Greenwich when the true distance of the centers of the Sun and Moon is c, calculated by Art. 433, $\Delta=PM$, the Moon's north polar distance, and $\Delta'=PS$, the Sun's. Then

$$\cos\phi=\frac{\cos(\mu+\sigma-n)-\cos c\,.\cos P}{\sin c\,.\sin P},$$

$$\cos\theta=\frac{\cos\Delta'-\cos\Delta\,.\cos c}{\sin\Delta\,.\sin c},$$

$$PMZ=180^0-(\theta+\phi),$$

$$\sin ZM=\frac{P}{p-\pi},$$

$$\cos PZ=\sin ZM\,.\sin\Delta\,.\cos(180^0-\theta-\phi)+\cos ZM.\cos\Delta,$$

which determines the terrestrial latitude of Z;

$$\text{and }\cos ZPM=\frac{\cos ZM-\cos PZ\,.\cos\Delta}{\sin PZ\,.\sin\Delta},$$

$$\cos MPS=\frac{\cos c-\cos\Delta\,.\cos\Delta'}{\sin\Delta\,.\sin\Delta'},$$

$$ZPS=ZPM+MPS;$$

therefore the terrestrial west longitude of the place whose zenith is Z

$= ZPS - GPS$

$= ZPS -$ the apparent time at Greenwich, is known.

In order to find all the places which see the different phases of the eclipse, P and c must be made to vary in the manner pointed out in Art. 439.

For actual calculation, the equations of this article may be simplified, and expedients adopted, by which the labour of finding many places where the eclipse is visible is much lessened*.

On the Number of Eclipses.

441. The Sun is more than a month in passing through the solar ecliptic limits; therefore there *must* necessarily be two eclipses of the Sun in a year, one at each node. If a solar eclipse happens very near the node, then at the preceding conjunction the Sun had not entered the ecliptic limit, and at the succeeding conjunction will have passed beyond it; therefore there *may* be only one eclipse at each node, or two solar eclipses in a year. If a solar eclipse happen very soon after the Sun has entered the ecliptic limits, there may be another eclipse near the time when it leaves the ecliptic limits, and as the same may happen at each node, there *may* be four solar eclipses in a year.

The center of the Earth's shadow is less than a month in passing through the lunar ecliptic limits; therefore there *can* be no more than one eclipse at each node, and there *may* be none; or there can be only two lunar eclipses in a year, and there *may* be none.

Hitherto the Moon's node has been considered stationary, but since the node recedes about 20° each year, if a solar or lunar eclipse happen near the beginning of the year, there may be one other of the same kind, at the *same* node, near the end of the year; therefore upon the whole there may be seven eclipses; five of the Sun and two of the Moon, or four of the Sun and three of the Moon, in the course of a year; and there may be only two, in which case they will be both solar eclipses.

On the Occultation of fixed stars by the Moon.

442. When the Moon passes between a fixed star and the Earth, the star is said to suffer an *occultation*. From this

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[*] See Delambre, *Astronomie*, Chap. xxvi; and the Appendix to the Nautical Almanac, for 1836.

description it is obvious, that an occultation is very similar in its physical circumstances to a solar eclipse, but is simpler, on account of the star being without motion, and unaffected by parallax. As in a solar eclipse, the times of occultation will be different at different places on account of parallax, and an occultation may happen at some places which has no existence at others. It has lately been suggested that, on account of the bending of the rays by which an occultation or solar eclipse is seen, these phenomena will appear as they would to a spectator considerably elevated above the Earth's surface; and that, consequently, Refraction may sensibly affect the times of these phenomena.

443. *To find the time of an occultation.*

From the known longitudes of the Moon and star, and the Moon's motion in longitude at the time when her longitude = that of the star, the time T of conjunction of the Moon and star may be found. Let l and λ be the difference of longitude and latitude of the two bodies for the time T, f and g the horary increments of l and λ at the same time, and s the latitude of the star; then, as in Art. 436. if t be the time from conjunction of first or last contact, S (fig. 72.) the star, and M the Moon's center, at the instant of first or last contact, and P the pole of the ecliptic,

$$SM = \mu,\ MN = \lambda + gt,\ NS = (l + ft) \,.\, \cos s\,;$$

$$\therefore\ \mu^2 = (\lambda + gt)^2 + (l + ft)^2 \,.\, \cos^2 s,$$

from which equation finding t and t', the two values of t, we obtain $T + t$ and $T + t'$, the times of the beginning and end of the occultation.

If λ and l are the differences of the geocentric longitudes and latitudes of the Moon and star, $T + t$ and $T + t'$ are the times at which the occultation would be observed from the center of the Earth; but if λ and l are the differences of the true latitude and longitude of the star, and of the apparent latitude and longitude of the Moon deduced from the Tabular values at the time T by applying the correction for parallax at the given place, the apparent times of the first and last contact at the place of observation are obtained.

CHAPTER XVII.

ON THE TRANSIT OF AN INFERIOR PLANET OVER THE SUN'S DISK; AND ON DOUBLE AND PERIODICAL STARS.

444. WHEN an inferior Planet is so near the node of its orbit at the time of inferior conjunction, that its latitude, as seen from the Earth, does not exceed the sum of its apparent semi-diameter and that of the Sun, it lies between the Sun and the Earth, and is seen to pass like a dark spot over the Sun's disk. Hence for a transit of Venus, the latitude at inferior conjunction must not exceed $961'' + 31''$ or $992''$; and consequently her distance from the node must not exceed $1^0. 50'$. In the case of Mercury the limiting latitude $= 961'' + 6'' = 967''$, and the consequent distance from the node $= 3^0. 28'$.

445. The transit of an inferior planet over the Sun's disk is exactly the same, mathematically considered, as a solar eclipse, and the same formulæ may be applied to calculate the one which were used for the other, by changing the quantities f, g, &c. which have reference to the Moon, into the corresponding quantities for the planet; therefore, if g be the motion of Venus in latitude, f the relative motion in longitude of Venus and the Sun, λ the latitude of Venus at the time of conjunction T, c the sum of the semi-diameters of the Sun and Venus, and $\tan n = \frac{g}{f}$,

the time when the transit begins

$$= T - \frac{\sin n}{g} . \{- \lambda \sin n + \sqrt{c^2 - \lambda^2 \cos^2 n}\},$$

the time when the transit ends

$$= T + \frac{\sin n}{g} . \{\lambda \sin n + \sqrt{c^2 - \lambda^2 \cos^2 n}\}.$$

Since c = the sum of the semi-diameters of the Sun and Venus, the times of first and last contact, as seen from the center of the Earth are determined, but if c = the semi-diameter of the Sun − the semi-diameter of Venus, the times of Venus having entered and of beginning to emerge from the Sun's disk, are obtained.

446. As in a solar eclipse, and in the way explained at Art. 105, spectators on the Earth's surface refer the planet to different parts of the Sun's disk; on which account the observed times of the first and last contact, and the whole duration of the same transit, are different in different places. This difference of duration arises from the difference of parallax of the Sun and Venus, and may be found by the formulæ of the preceding Chapter, when the difference of parallax is given; therefore, conversely, when the difference of the times of duration is known by observation, the difference between the two parallaxes may be obtained.

447. The ratio of the parallaxes of Venus and the Sun is known from the ratio of their distances from the Earth at the time of transit; therefore, if the difference of the parallaxes be found from the observed durations of the same transit in different places, the Sun's parallax may be determined. On account of the accuracy with which the entrance of a dark body like Venus upon the Sun's disk may be observed, the duration of a transit may be determined with great exactness, which renders this by far the best method of finding the Sun's parallax. We may judge of its exactness, by considering that the greatest difference in the durations of the transit of 1769 observed at different places, was at least 18^m; therefore, as the Sun's parallax is about $8'',57$, and is proportional to the difference of the durations, as will be seen below, it would require an error of $18^m \div 8.57$, or more than 2 minutes, in the difference of the observed durations, to produce an error of $1''$ in the computed parallax of the Sun. As it is highly probable that the difference of the durations was known within 15^s, therefore, it is in the same degree probable that the Sun's parallax was determined within $0'',25$ of its true value. If the observed difference, which was possible, had been 25^m, it would have required an error of nearly 3^m in the difference of the durations to produce an error of $1''$ in the computed solar parallax.

The superior advantages of a Transit of Venus for finding the solar parallax above one of Mercury, consist, first in the greater difference of parallax of the Sun and Venus = 23″ at the time of inferior conjunction, whilst for Mercury this difference is only about 9″; and 2nd in the slower relative motion of Venus and the Sun during a transit, = 234″ nearly, whilst that of Mercury and the Sun is 550″; if there was a Planet between Venus and the Earth, it would afford still greater facilities for finding the Sun's parallax.

448. *To find what places are most eligible for observing a transit of Venus.*

Since the error in the computed parallax of the Sun diminishes as the difference of the observed times of transit increases, it becomes important to enquire at what places it is most advantageous to make the observations. Not to enter into exact calculations, which may be made in a manner similar to those of Art. 440, we may determine the most eligible places with sufficient accuracy by means of a common globe; and this we shall exemplify by taking the instance of the transit of 1769. Suppose (as was nearly the case in that transit) that the transit begins at 7 o'clock, apparent Greenwich time, and continues 6 hours, and that the Sun's declination at the same time is 22^0 north. Place the globe so that the altitude of the north pole may equal 22^0, and turn it till the places in west longitude $7.15^0 = 105^0$ are brought under the fixed meridian; then, in this position of the globe, the Sun will be visible at all places above the horizon of the globe, and will be in the zenith of a place in north latitude 22^0, at the beginning of the transit. If now the globe be turned in the same direction through $6.15^0 = 90^0$, all places above the horizon in the second position will see the Sun at the time of last contact, and those places which have continued above the horizon whilst the globe was turned, will see the whole of the transit.

In the first position of the globe the commencement of the transit is not seen by all places in the horizon, but only at one place, which lies in the great circle passing through the centers of the Sun and Venus, which place may be determined as in Art. 439.

The place which has the Sun in its zenith at the beginning of the transit, sees the first contact on the Sun's eastern limb nearly at the time when it would be seen from the center of the Earth; but as this place is near the horizon when the transit ends, the duration is diminished. Since the latitude of Venus is north, at places in higher northern terrestrial latitudes than 22°, Venus is depressed by parallax towards the Sun's center, and the duration of the transit is prolonged and this effect is increased with the north latitude of the observer. For the same reason, the duration is diminished at places in south latitude; therefore, among all accessible places which are above the horizon in the two positions, those ought to be chosen which are most north or south, in order that the observed difference of duration may be the greatest possible.

Places in high north latitudes, where the nights are short, may see the first contact before sun-set, and the last after sunrise, and in this case also, the duration of the transit would be increased.

449. *To find when a transit of Venus may be expected.*

A transit can only happen when Venus is near its node at the time of conjunction; if, therefore, the Earth and Venus are together in the same node at any time, when they return to the same position again, each of them must have made a number of complete revolutions. Let n be the number of revolutions made by Venus, m the number made by the Earth in the interval between two conjunctions; then, since the Earth revolves in $365^d,256$, and Venus in $224^d,7$, we must have

$$m \, . \, 365^d,256 = n \, . \, 224^d,7 \, ; \quad \therefore \frac{m}{n} = \frac{224,7}{365,256},$$

the converging fractions approximating to which, are

$$\frac{8}{13}, \; \frac{235}{382}, \; \frac{713}{1159}, \; \&c.$$

Therefore, a transit may be expected at the same node after an interval of 8 or 235 or 713 years. Now a transit happened at the descending node in 1769, and at the ascending node in 1639; therefore, transits may be again expected at the descending node in

2004, 2012, 2247, 2255, 2490, 2498, 2733;

and at the ascending node in

1874, 1882, 2117, 2125, 2360, 2368, 2603.

To ascertain whether there will be a transit at these times, a more exact investigation must be made, and the transit will happen or not, according as the least distance of the centers of the Sun and Venus is less or greater than the sum of their semi-diameters. By reason of the present position of Venus's node, transits can only happen in June or December; the next will be 1874 Dec. 8, 1882 Dec. 6, 2004 June 7, 2012 June 5.

450. *To find the Sun's parallax from the difference of the times of duration of the same transit, observed at different places.*

Let T, T' be the *Greenwich* times when the first and last contact would be seen from the center of the Earth; $T+t$, $T'+t'$ the *Greenwich* times of the first and last contact, seen from the place of observation; l, l' the *true* longitudes of Venus and the Sun at the time T; ϕ the horizontal parallax of Venus, and f, g, &c. as before; then $a\phi$ will be the effect of parallax on the longitude of Venus, where a is a function of the position of Venus, and since Venus and the Sun are nearly coincident in position $a\pi$ will be the effect of parallax on the Sun's center, so that the *apparent* difference of longitude at the time T

$$= l - l' + a(\phi - \pi);$$

therefore, the *apparent* difference of longitude at the time $T+t$,

$$= l - l' + a.(\phi - \pi) + ft.$$

Similarly, the apparent latitude of Venus at the time $T+t$

$$= \lambda + b.(\phi - \pi) + gt.$$

Therefore, since the transit appears to begin at the time $T+t$, if c be the sum of the semi-diameters of the Sun and Venus,

$$c^2 = \{l - l' + a.(\phi - \pi) + ft\}^2 + \{\lambda + b.(\phi - \pi) + gt\}^2$$

$$= (l-l')^2 + \lambda^2 + 2.\{a.(l-l') + b\lambda\}.(\phi - \pi) + 2.\{f.(l-l') + g\lambda\}t;$$

neglecting the squares and products of the very small quantities t and $\phi - \pi$. But, since l, l', λ, are the longitudes and latitudes at the time the transit begins to a spectator in the Earth's center,

$$c^2 = (l - l')^2 + \lambda^2;$$

$$\therefore \{a . (l - l') + b\lambda\} . (\phi - \pi) + \{f . (l - l') + g\lambda\} t = 0;$$

$$\therefore t = -\frac{a . (l - l') + b\lambda}{f . (l - l') + g\lambda} . (\phi - \pi)$$

$$= \alpha . (\phi - \pi), \text{ by substitution.}$$

Therefore, the Greenwich time of the first contact being seen at the place of observation

$$= T + \alpha . (\phi - \pi).$$

If β be the corresponding quantity to α for the time T', the time of the last contact

$$= T' + \beta . (\phi - \pi);$$

therefore, the whole duration (D)

$$= T' - T + (\beta - \alpha) . (\phi - \pi).$$

If D' be the duration observed at any other place, and β', α' corresponding values of β and α,

$$D' = T' - T + (\beta' - \alpha') . (\phi - \pi);$$

$$\therefore D' - D = (\alpha - \alpha' - \beta + \beta') . (\phi - \pi),$$

$$\therefore \phi - \pi = \frac{D' - D}{\alpha - \alpha' - \beta + \beta'}.$$

Now, $\phi : \pi ::$ the distance of the Sun from the Earth: the distance of Venus; therefore $\phi - \pi : \pi ::$ the distance of Venus from the Sun : the distance of Venus from the Earth. Let this ratio, which is known, be that of $1 : e$;

$$\therefore \pi = e . (\phi - \pi) = e . \frac{D' - D}{\alpha - \alpha' - \beta + \beta'}.$$

The transits of 1761 and 1769 were observed in various places, and the most probable value of π, deduced from the different times of duration, has been found to be $8'',57116$.

451. The transits of Mercury, which by reason of his more frequent returns to his node, occur oftener than those of Venus, may be used in the same manner for determining the Sun's parallax; but, as has been said, on account of his quick motion and the smallness of his parallax the observations are subject to larger errors, and no great dependence can be placed on the results. For many ages the transits of Mercury will happen only in May and November; the next which will be visible in Europe will happen 1845 May 8, 1848 Nov. 9, 1861 Nov. 11, 1868 Nov. 4, 1878 May 6, 1894 Nov. 10.

Double and Periodical Stars.

452. Several of the stars when examined with powerful telescopes are found to be double, that is, to consist of two or more stars placed close together. In some instances this is probably only an appearance occasioned by two or more stars being situated nearly in the same straight line passing through the Sun; in such cases the stars are only optically double. But that many are physically double, that is, that they are systems revolving about their common center of gravity and subject to the same dynamical laws as the solar system, we are lead to suppose by considering the immense number of such combinations, (nearly 6000 having been observed), the closeness of the stars in each, and the nearly equal magnitude of the individuals in several, and the large proportion among them of stars of considerable magnitude; all which circumstances are inconsistent with the supposition that the proximity to one another of double stars is only apparent or accidental.

453. This hypothesis is fully borne out by observation; for if the angular distance of two stars of a binary combination from one another, and the angle, called their angle of position, which their line of junction makes with the declination circle passing through the central star, be observed from time to time, the changes which these quantities undergo agree exactly with the supposition that one star describes an orbit about the other. Sometimes the angle of position increases continually in the same direction, whilst the distance undergoes little variation; this happens when the orbit of the satellite is nearly circular, and its plane at right angles to the line joining the

central body and the Sun. When on the other hand the plane of the orbit nearly passes through the Sun, so that the apparent motion of the satellite is nearly in a straight line, the angle of position undergoes little change, whilst the distance continually increases or diminishes, and sometimes becomes zero; in which latter case the satellite suffers an occultation from its primary, as for example τ Ophiuchi, which in 1781 was a double star, but afterwards appeared perfectly round, and is now again double.

454. Of these binary systems, one star is generally much smaller than the other, as in the pole star where one is of the 2nd and the other of the 11th magnitude; but in some, the stars are nearly equal, as in γ Arietis where both are of the 5th magnitude, in Castor where one is of the 3rd and the other of the 4th magnitude, and in 61 Cygni where both are of the 5th. Another curious circumstance attending several of the double stars is that they exhibit the phenomenon of contrasted or complementary colours; the larger star being of a ruddy or orange hue whilst the smaller one appears blue or green.

From observations of the distances and angles of position at different epochs, the elliptic elements of the orbits of some of the binary stars have been computed; according to Mädler the chief results obtained are the following*:

Names of Stars.	Period of Revolution.	Semi-axis major of Ellipse.	Eccentricity.
Castor	$230^y.299$	$7''.008$	0.7972
61 Cygni . .	452	15.43	——
γ Virginis . .	157.5623	5.35	0.86805
ξ Ursæ . . .	60.4596	2.29	0.4037
σ Coronæ . .	199.95	2.9279	0.5768
70 Ophiuchi .	80.61	4.3159	0.4772
γ Leonis . .	1200		
η Coronæ . .	43.34	1.1912	0.3537
ζ Cancri . . .	55		

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* Schumacher's Jahrbuch for 1839.

455. Double stars require excellent telescopes for their observation, being in general so close to one another as to make the use of very high magnifiers necessary to separate them. It is only instruments of the very best description that will exhibit a sensible interval between the stars in some of the systems just mentioned; viz. γ Virginis where the present angular dist. $\Delta = 1''$, and both stars are of the 3rd magnitude; and η Coronæ where $\Delta = 1''$ and the magnitudes are 5 and 6. For Castor where $\Delta = 5''$, and the magnitudes are 3 and 4, and for ζ Cancri, a treble star in which the values of Δ are $1'',13$ and $5'',7$, and the magnitudes 5 and 6, instruments not quite so powerful will suffice. The double stars ϵ Lyræ and 5 Lyræ, from their close proximity, are supposed to form a physical system of four stars.

456. Many of the stars, as was stated, have proper motions. This is true not only of single stars, but also of double stars, which, besides revolving about their common center of gravity, are transferred without separating from one another by a progressive motion common to both; and in the case of the latter this furnishes a strong proof that their motions are not independent. Of the single stars μ Cassiopeiæ has the greatest proper motion yet observed, amounting to an annual displacement of $3'',74$, or to a secular motion of $571''$ in R.A. and $150''$ in declin.; and the double star 61 Cygni has an annual displacement of $5'',3$, or a secular motion of $503''$ in R.A. and $339''$ in dec., the two stars of which it is composed retaining the same distance from one another of $16''$. This star is probably one of the nearest of all the fixed stars to the Earth; and Bessel thinks he has determined its annual parallax to be about $0'',3136$, and its consequent distance from the Sun about 657700 radii of the Earth's orbit. The apparent directions of these proper motions are various, and do not seem to have a common tendency to one determined point of the heavens; as ought to be the case if they are solely due to a progressive motion of the solar system, for they would in that case all tend to the point which the solar system was leaving.

457. Periodical stars are those which, though not distinguished from other stars by any apparent change of po-

sition, undergo a regular periodical change of brightness, amounting in some cases to a total extinction and recovery of light. The most remarkable of these is the star called Algol or β Persei. It usually appears as a star of the 2nd magnitude, which it continues to be for about 61^h; it then suddenly begins to diminish in brightness, and in about 4^h is reduced to the 4th magnitude, in which state it continues for about 18^m without sensible alteration. It then begins again to increase, and in $4^h\ 40^m$ more is restored to its usual brightness, going through all its changes in 2.8673 days. These appearances have been supposed to be owing to the revolution round Algol of some large opaque body which, when interposed between us and Algol, intercepts a portion of its light. Another remarkable periodical star is *o* Ceti which oscillates between the 2nd magnitude and total invisibility; it remains at its greatest brightness about 26^d, and invisible about 3 months, and runs through all its phases in 331.96 days. In all the periodical stars the phase of least light continues much longer than that of greatest light; the brightness does not increase and diminish by the same gradations; and both the extent of the changes, and the duration after which they recur, appear to be affected by physical causes of which we are at present ignorant.

CHAPTER XVIII.

TERRESTRIAL LONGITUDE.

458. If PZp, $PZ'p$ (fig. 74.) be the meridians of two places, whose zeniths are Z and Z', and PSp the declination circle passing through the Sun's center; ZPZ' the difference of longitude of the places $= ZPS - Z'PS =$ the difference of the apparent times at the two places, expressed in degrees. In the same manner it may be shewn, if S be the mean Sun or the first point of Aries, that the difference of longitude of two places = the difference of the mean or sidereal times at the same places, expressed in degrees. Therefore, the problem of finding the difference of longitude reduces itself to finding the difference of the reckoned times, whether solar, sidereal, or mean-solar, for the same instant of absolute time, at the places for which the difference of longitude is required.

If PZp be the meridian of Greenwich, which the English take for their *first meridian*, ZPZ' is the west longitude of Z'; therefore, the longitude of any place = the time at Greenwich − the time at that place. If the time at the place of observation is greater than the time at Greenwich, the longitude of the place is measured east of Greenwich.

459. *To find the longitude of a place by a chronometer.*

The simplest method of finding the longitude is by a chronometer which can be depended on. If a person with such a chronometer, which points out and preserves the time under the meridian of Greenwich, were to travel to a place under a different meridian, the chronometer would continue to denote the Greenwich time nearly; therefore, finding the time under the second meridian by some of the methods of Chap. VI, and comparing it with the corrected time shewn by the watch, the difference will be the longitude of the place of observation in time, east when the time shewn by the watch is the less, and west when greater.

On account of the perfection with which these instruments are now made, this method is extremely valuable; but it is obvious that its value must depend in a great measure upon our having it in our power to ascertain if the watch goes uniformly, and upon our being able to determine and allow for any errors in its rate of going; that is, upon our being able to find at *any* place, by independent methods, the time under the first meridian, or the longitude of the place of observation.

460. The difference of longitude of two places may be determined astronomically, either by observing some phenomenon which is seen from both places at the same instant, such as a lunar eclipse, an eclipse of one of Jupiter's satellites, signals made on the Earth by the bursting of rockets, or the explosion of gunpowder; or else, by observing phenomena, not themselves happening for both places at the same instant, but from which the times of happening of some other phenomenon, which is independent of the places of observation, and therefore the same for both, may be deduced. Such phenomena are solar eclipses, occultations of fixed stars by the Moon, the observation of stars that culminate with the Moon, and the observed distance of the Moon from the Sun, or a fixed star, or planet.

461. *Longitude by lunar eclipses.*

A lunar eclipse, which is visible to half the Earth, would afford an excellent method of determining the longitude of places, if it could be observed with sufficient accuracy; because, the eclipse happening for both places at the same instant of absolute time, the difference of the reckoned times at two places would give their difference of longitude. But, on account of the difficulty of determining the separation of the shadow and penumbra, two observers of an eclipse at the same place may differ more than 2^m in noting the time of contact; therefore, the longitude determined from the observed times of contact may be erroneous to the amount of 4^m of time, which produces an error of 1^0 of longitude. The error may be diminished, by observing at the two places the times when the shadow reaches known spots on the Moon's disk, determining the longitude from each pair of observations of the same spot, and taking the mean for the true longitude.

This method, even if it were more accurate, must be of little use at sea, on account of the comparative rarity with which lunar eclipses recur.

462. *Longitude by the eclipses of Jupiter's satellites.*

Although not so accurate as some of the succeeding methods, yet, on account of the frequency with which the eclipses recur, and the easiness of the observations, this method of finding the longitude for great distances is often used. In order to facilitate its application, the times at which the eclipses occur at Greenwich are given annually in the Nautical Almanac, three or four years in advance, by comparing which, with the observed times at any other place, the longitude of the place may be determined. The method cannot be used at sea on account of the difficulty of directing a telescope properly on shipboard, which, as yet, has been found insurmountable; and even on land at the same station, two observers with telescopes of different magnifying powers and apertures, will seldom agree within a second of each other. Indeed, as the satellite has a considerable apparent diameter as seen from Jupiter's center, and as the penumbra extends to a sensible although very small distance beyond the shadow, the instant of the total immersion of the satellite, or of its complete emersion, cannot be exactly determined; and therefore the only case in which this method can be of much use in determining terrestrial longitudes is where both the immersion and emersion are observed with the same telescope and by the same person, and by that means the precise instant of the satellite's opposition to the Sun becomes known.

463. *Longitude by signals.*

Of all methods of determining small differences of longitude, perhaps the most accurate is that by signals made on the Earth's surface. The bursting of a rocket, or the explosion of gunpowder, fired from an elevated spot, on a clear night, may be seen distinctly with a telescope at a distance varying from twenty to ninety miles, according to the power of the instrument; therefore, by observing the times at which the same explosion is seen at two places, the difference of longitude of the places may be determined; for the absolute time of the explosion will be the same at both stations within a quantity depending upon the velocity of light not appreciable in a difference of distance of the magnitude here supposed. If the places whose difference of longitude is required, are too distant to observe the same signal, a series of such signals must be made and noted by observers placed at intermediate stations from which two adjacent signals can be seen. Here, as in other cases where accuracy is required, we must not be contented with the longitude determined from one observation, but take the mean of several longitudes determined by numerous observations.

The times at which the explosions are to take place, must be previously agreed on by the observers, and the telescopes must be properly directed during the day-time. Great attention also must be paid to the rate of the chronometers, and to the determination of the time at the places of observation, which is done by observing the Sun's transit, or by corresponding heights of the Sun during the preceding day, and also of a fixed star near the time of observation.

464. *Longitude by the method of Moon-culminating Stars.*

Of the second class of methods by which the longitude is determined, the simplest is by finding the increase of the Moon's right ascension in the interval between the transit of the Moon over the first meridian, and over the meridian whose longitude is required. This is done by observing the differences of the right ascensions of the Moon's bright limb and a star nearly in the Moon's parallel of declination and not differing much from her in right ascension, at the two meridians; then, supposing the rate of increase of the Moon's right ascension to be known, the difference of longitude may be obtained.

Let T be the time by the clock when the Moon's bright limb is on the meridian of any place, t the time of transit of a known fixed star, $24 + x$ the interval between two successive transits of the same star, x being the error of the clock in that interval; then $24 + x : T - t :: 360^0$: the difference of right ascension of the Moon's bright limb and the star at the instant the limb was on the meridian; to which, adding the known right ascension of the star, the right ascension of the Moon's bright limb (a') when on the meridian is determined. Now, the stars proper to be observed, and the right ascension of the

Moon's bright limb when on the meridian of Greenwich are given in the Nautical Almanac for every day of the year, from whence the daily increment of right ascension may be determined; let, therefore, a be the right ascension of the Moon's bright limb when on the meridian of Greenwich, e the increment of right ascension in the time between two successive transits over the same meridian; then, whilst the Moon, by its relative motion, separates from the meridian of Greenwich by an angle of 360^0, its real motion in right ascension is e, and whilst it separates by an angle equal the difference of longitude, the motion in right ascension is $a - a'$, therefore considering the change of right ascension uniform,

$$\text{the required longitude} = \frac{a - a'}{e} \cdot 360^0.$$

When great accuracy is wanted, the difference of longitude corresponding to the increase of right ascension $a - a'$ must be determined by interpolation. This is considered one of the best modes of determining the longitude of distant places when the observer, furnished with a transit instrument, can obtain a landing.

465. A method similar in principle to the preceding, but much more subject to error, is, by calculating the hour angle of the Moon from a single observed altitude, an approximate declination, and the latitude of the place; then, the Moon's right ascension = the Sun's right ascension + the Sun's hour angle + the Moon's hour angle, is known. Having found the right ascension of the Moon, it may be compared with the right ascension when on the meridian of Greenwich, and the difference will be the increase of the Moon's right ascension, whilst by the diurnal motion she describes an angle = the longitude of the place + the Moon's hour angle: this angle may be found exactly as in the preceding Article, and subtracting from it the Moon's hour angle, we find the longitude of the place. We have supposed Greenwich and the Sun to the west, and the Moon to the east of the meridian in question.

466. When the difference of longitude of two places is nearly known, a more correct value may be obtained by observing the Moon's transits over the meridian in the following manner.

Let t, t' be the observed differences at two stations A, A' (whose east longitudes from the meridian of Greenwich are known very nearly, that of A' being the greater,) of the times of transit of the Moon's western limb, and a known star situated west of the Moon, and having nearly the same declination, expressed in sidereal time; also, let c, c' be the apparent times, as shewn at Greenwich, of the Moon's center being on the meridians of A, A': and a, δ, d; a', δ', d', the right ascension, declination, and angular radius of the Moon as they would appear to an observer at the Earth's center at those times; therefore, the true observed motion in right ascension of the Moon's center between the observations, expressed in sidereal time,

$$= \Delta = t - t' + \frac{1}{15}\left(\frac{d}{\cos\delta} - \frac{d'}{\cos\delta'}\right).$$

Now $c - c'$ is an interval of *apparent* time; but if s be the increase of the Sun's right ascension, on the day of observation, in 24^{h}, expressed in sidereal seconds, the equivalent sidereal interval is $\frac{86400 + s}{86400}(c - c')$, and in this interval the true motion of the Moon's center in right ascension is $\frac{1}{15}(a - a')$. Also the time between the Moon's limb being upon the two meridians is $x + \Delta$, if x be the difference of longitude in sidereal seconds, in which time the true motion in right ascension of the Moon's center is Δ;

$$\therefore \frac{86400 + s}{86400}(c - c') : x + \Delta :: \frac{1}{15}(a - a') : \Delta;$$

$$\therefore x = \Delta\left\{\frac{86400 + s}{5760} \cdot \frac{c - c'}{a - a'} - 1\right\}.$$

This method is independent of the errors of the lunar tables, except so far as the horary motion in right ascension of the Moon is concerned; it does not require the position of the star to be correctly known; and an error of several seconds in the state of the clock is of no consequence*. The stars employed in this method are those as near the Moon as possible, at the time of observation.

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* Vid. Astr. Nachr. N°. 26, and Memoirs of Astronomical Society, Vol. II.

467. To find the longitude from the observed angular distances between the centers of the Moon and Sun or other heavenly bodies.

Let A, a, δ be the true altitudes, and the true distance of the centers of the Moon, and of the Sun (or some other heavenly body); A', a', δ' the observed values of the same quantities, at a time which is exactly known, the rate of the chronometer having been determined by nearly contemporary observations. Then, if P and R be the Moon's parallax and refraction, p and r the Sun's, calculated from the apparent altitudes A' and a' by the formulæ of Chap. IX,

$$A = A' - R + P, \quad a = a' - r + p.$$

Let s and m (fig. 75.) be the apparent places of the centers of the Sun and Moon, S and M the true places, Z the zenith, and let $x =$ the angle SZM; then, from the triangle SZM,

$$\cos\delta = \sin A \,.\, \sin a + \cos A \,.\, \cos a \,.\, \cos x$$

$$= \sin A \sin a + \cos A \,.\, \cos a \,.\, (2\cos^2 \tfrac{1}{2}x - 1)$$

$$= -\cos(A + a) + 2\cos A \,.\, \cos a \cos^2\tfrac{1}{2}x.$$

But $\cos\delta = 1 - 2\sin^2\frac{1}{2}\delta$, $\cos(A + a) = 2\cos^2\frac{1}{2}(A + a) - 1$;

$$\therefore \sin^2\tfrac{1}{2}\delta = \cos^2\tfrac{1}{2}(A + a) - \cos A \,.\, \cos a \,.\, \cos^2\tfrac{1}{2}x.$$

And from the triangle Zsm,

$$\cos^2\tfrac{1}{2}x = \frac{\sin\frac{1}{2}(Zm + Zs + ms) \,.\, \sin\frac{1}{2}(Zm + Zs - ms)}{\sin Zm \,.\, \sin Zs}$$

$$= \frac{\cos\frac{1}{2}(A' + a' - \delta') \,.\, \cos\frac{1}{2}(A' + a' + \delta')}{\cos A' \,.\, \cos a'};$$

$$\therefore \sin^2\tfrac{1}{2}\delta = \cos^2\tfrac{1}{2}(A + a) -$$

$$\frac{\cos A \,.\, \cos a}{\cos A' \,.\, \cos a'} \,.\, \cos\tfrac{1}{2}(A' + a' - \delta') \,.\, \cos\tfrac{1}{2}(A' + a' + \delta').$$

Let the subsidiary angle ϕ be determined from the equation

$$\sin\phi = \sqrt{\left\{\frac{\cos A \,.\, \cos a}{\cos A' \,.\, \cos a'} \,.\, \frac{\cos\frac{1}{2}(A'+a'-\delta') \,.\, \cos\frac{1}{2}(A'+a'+\delta')}{\cos^2\frac{1}{2}(A+a)}\right\}};$$

$$\therefore\ \sin^2\tfrac{1}{2}\delta = \cos^2\tfrac{1}{2}(A+a) - \cos^2\tfrac{1}{2}(A+a) \,.\, \sin^2\phi;$$

$$\therefore\ \sin\tfrac{1}{2}\delta = \cos\tfrac{1}{2}(A+a) \,.\, \cos\phi,$$

from which equation δ, the true distance of the centers of the Sun and Moon, may be found.

The angular distances between the centers of the Moon and certain heavenly bodies, such as they would appear to an observer at the center of the Earth, are registered in the Nautical Almanac for every third hour of Greenwich Mean Time. When a lunar distance has been observed on the surface of the Earth and reduced to the center by clearing it of the effects of parallax and refraction by the foregoing process, it will be found to lie between two successive registered values; and the exact Greenwich Mean Time corresponding to the observed distance must be determined in the usual way by interpolation, or by adding the proportional part of 3^h to the time answering to the nearest registered distance; and taking the difference between this and the apparent time at the place of observation already noted, we obtain the longitude of the place of observation in time.

468. Among the various ways by which the Moon's true distance (δ) is found, from the apparent distance and the apparent altitudes of the Sun and Moon, the preceding, by *Borda*, is considered the best. The following method is also one of great simplicity, as the latter part of the operation, with the assistance of a table of versines, is performed by simple addition and subtraction, without requiring logarithms;

$$\because\ \cos\delta = \sin A \,.\, \sin a + \cos A \,.\, \cos a \,.\, \cos x;$$

$$\therefore\ \cos\delta = \sin A \,.\, \sin a + \cos A \,.\, \cos a\,(1 - 2\sin^2\tfrac{1}{2}x)$$

$$= \cos(A-a) - 2\cos A \,.\, \cos a \,.\, \sin^2\tfrac{1}{2}x,$$

$$= \cos(A-a) - \frac{2\cos A \,.\, \cos a}{\cos A' \cos a'} \,.\, \sin\tfrac{1}{2}(A'+\delta'-a') \,.\, \sin\tfrac{1}{2}(a'+\delta'-A').$$

$$\text{Let } \cos\theta = \tfrac{1}{2}\,\frac{\cos A \,.\, \cos a}{\cos A' \cos a'};$$

$$\begin{aligned}
\therefore\ \cos\delta &= \cos(A-a) - 4\cos\theta \,.\, \sin\tfrac{1}{2}(A' + \delta' - a') \,.\, \sin\tfrac{1}{2}(a' + d' - A') \\
&= \cos(A-a) + 2\cos\theta \,.\, \{\cos\delta' - \cos(A' - a')\} \\
&= \cos(A-a) + \cos(\theta + \delta') \\
&\qquad + \cos(\theta - \delta') - \cos(\theta + A' - a') - \cos(\theta + a' - A');
\end{aligned}$$

$$\begin{aligned}
\therefore\ \text{ver-sin}\,\delta = \text{ver-sin}\,(A-a) &+ \text{ver-sin}\,(\delta' + \theta) + \text{ver-sin}\,(\delta' - \theta) \\
&- \text{ver-sin}\,(A' - a' + \theta) - \text{ver-sin}\,(A' - a' - \theta).
\end{aligned}$$

469. The heavenly bodies, whose distances from the Moon at any time are computed in the Nautical Almanac, are those which lie at a distance from the Moon's place at that time not exceeding 120°. In making the observation that body is to be preferred, towards or from which the velocity of the Moon is the greatest (i. e. the difference of their declinations should be as small as possible compared with the difference of R. A.), because then the greatest reliance can be placed on an observation of the distance; but every registered object should be observed, if opportunity offer, in order to get the most accurate value for the Greenwich mean time of the observation. The method of finding the Longitude by Lunar Distances is the only one which is of much use at sea, as it requires no other instrument than the sextant, and the observations are such as can be very frequently made. There ought to be three persons to observe at the same time the two altitudes, and the apparent distance. But where there is only one observer, it will be sufficient to observe the one altitude, then the distance, and then the other altitude; and immediately to repeat the operation of observing the altitudes, and from these observed altitudes, and the known intervals between the observations, to find, by taking the proportional parts, the altitudes which would have been observed at the instant the distance between the centers was measured.

470. *To find the longitude from an observed occultation of a fixed star by the Moon.*

When this method is used, the longitude is supposed known, very nearly. If it were known exactly, and the Lunar

Tables were perfectly correct, then from the observed time of contact the corresponding Greenwich time would be known; and the longitude and latitude of the Moon's center for that time being taken out of the Tables and corrected for parallax, would be the apparent values of those quantities at the instant of the observation, and would therefore satisfy the condition of the distance of the Moon's center from the star being exactly equal to the Moon's semi-diameter. This consideration enables us to correct the assumed longitude of an observatory, and the errors of the lunar tables, as follows. Let θ be the approximate longitude in time, $\theta + t$ the true longitude, and let T be the reckoned time of the contact at the place of observation; therefore, $T + \theta + t$ is the corresponding Greenwich time. Let the Moon's true latitude and longitude, taken out of the tables for the time $T + \theta$, be corrected for parallax, and let l and λ be the resulting apparent longitude and latitude of the Moon's center at the Greenwich time $T + \theta$, and f and g the horary increments of the same quantities; also, let l' and λ' be the latitude and longitude of the star; then, at the time of contact,

the difference of longitude $= l - l' + ft$,

.... latitude $= \lambda - \lambda' + gt$:

therefore, if μ be the semi-diameter of the Moon,

$$\mu^2 = (l - l' + ft)^2 \cos^2\lambda + (\lambda - \lambda' + gt)^2$$
$$= (l - l')^2 \cos^2\lambda + (\lambda - \lambda')^2 + 2\{f.(l - l').\cos^2\lambda + g.(\lambda - \lambda')\}t,$$

neglecting the square of t;

$$\therefore\ t = \frac{\mu^2 - (\lambda - \lambda')^2 - (l - l')^2 \cos^2\lambda}{2.\{g(\lambda - \lambda') + f.(l - l')\cos^2\lambda\}},$$

from whence the correct longitude $\theta + t$ is known.

471. The preceding solution has been obtained upon the supposition that the lunar tables are correct; but as this is not often the case, the values of λ and l will be subject to slight errors, let these be δl and $\delta\lambda$; then, we have

$$\mu^2 = (l + \delta l - l' + ft)^2.\cos^2\lambda + (\lambda + \delta\lambda - \lambda' + gt)^2$$
$$= (l - l')\cos^2\lambda + (\lambda - \lambda')^2 + 2\{f.(l - l').\cos^2\lambda + g(\lambda - \lambda')\}t$$
$$+ 2(l - l')\cos^2\lambda.\delta l + 2(\lambda - \lambda')\delta\lambda,$$

neglecting the squares and products of δl, $\delta\lambda$, and t.

This equation may be put under the form

$$A + Bt + C\delta l + D\delta\lambda = 0,$$

and if the same occultation be observed at two other places, whose longitudes are known, we shall be furnished with two more simple equations of the form

$$A' + C'\delta l + D\delta\lambda = 0,$$

and from these three equations the values of t, δl, and $\delta\lambda$, may be found.

472. When extreme accuracy is required the apparent distance of the centers of the Moon and the occulted body may be calculated as follows:

Let S' (fig. 29. Plate III.) be the apparent place, and S the place as it would appear to an observer at the Earth's center, of the center of the Moon at the time of observation; then $PS = \Delta$, and $ZPS = h =$ tabular R.A. of Moon's center − R.A. of zenith, are given in the Nautical Almanac, the Greenwich mean solar time of observation being known. Also let $PS' = \Delta'$, $ZPS' = h'$, and $ZS' = z'$, be the apparent N.P.D. hour angle, and distance from the geocentric zenith, of the Moon's center; $ZP = c$, and $ZS = z$; then from the triangle ZSP, $\angle S$ and $\angle Z$ may be computed in the usual way. For calculating the apparent zenith distance and hour angle of the Moon's center, we have (Art. 249,)

$$\sin(z' - z) = \sin P \sin z',$$

$$\sin(h' - h) = \frac{\sin P \sin c \sin h'}{\sin\Delta},$$

from which z' and h' must be obtained by successive approximations; $\sin P$ being sin of hor. parallax

$$= \text{sin of hor. equat. parallax} \times \sqrt{1 - e^2\cos^2 c}, \text{ (Art. 248).}$$

Then apparent semi-diameter = geocentric semi-diameter $\dfrac{\sin z'}{\sin z}$.

$$\text{Also } \sin(\Delta' - \Delta) = \sin(z' - z)\cos S'$$

$$= \sin(z' - z)\cos(S - n) \text{ suppose;}$$

then $\cos(S - n) = \cos S \cos(h' - h) + \sin S \cos\Delta \sin(h' - h)$;

$$\therefore\ n = (h' - h)\cos\Delta - \tfrac{1}{2}(h' - h)^2 \cot S \sin^2\Delta \quad \text{(Art. 161)};$$

which determines n; and then Δ' the apparent N.P.D. becomes known.

Let δ' = apparent N.P.D. of the occulted body, d = difference between Δ' and δ', and a = the difference of the apparent right ascensions; then if $a\sqrt{\sin\delta' \sin\Delta'} = d\tan\phi$,

$$\text{apparent distance of centers} = d\sec\phi.$$

This compared with the observed distance of the centers will serve to test the accuracy of the tables, and the longitude of the place of observation.*

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* See the Cambridge Observations for 1837, p. lxxvii.

APPENDIX I.

ON THE PROJECTION OF THE SPHERE.

1. The projection of the sphere is a representation of the circles of the sphere on a plane given in position, as they appear to the eye situated at a given point; the investigation of the means of effecting this, is Perspective confined to these particular objects, and the use of it, the construction of Maps and Charts.

The plane on which the circles are to be delineated is called the primitive plane, and is most frequently supposed to be situated between them and the eye; if lines be drawn from all the points of the circles to the eye, these will by their intersection with the primitive, form the projection; that is, a representation which produces the same effect on the organ of vision as the original does.

2. We shall treat of three methods of projecting the sphere arising from the different positions of the eye, viz. the *orthographic*, *stereographic*, and *gnomonic*. In the first, the eye is supposed to be infinitely distant, so that the lines of vision are parallel to one another and perpendicular to the primitive; in the second, the eye is situated on the surface of the sphere, and the primitive passes through the center so as to have the eye in its pole; in the third, the eye is in the center, and the primitive a tangent plane to the sphere; and on which ever of these suppositions the projection is made, it is further termed *equatoreal*, *meridianal*, or *horizontal*, according as the plane of projection coincides with, or is parallel to the equator, or to the meridian, or horizon of any place.

3. The orthographic projection of any straight line = the line × cosine of its inclination to the primitive.

Let AB be any line (fig. 76.), DE the intersection with the primitive of a plane through AB perpendicular to the primitive; draw Aa, Bb perpendicular to DE, then ab is the projection of AB, and, drawing AC parallel to DE,

$$AC = AB \,.\, \cos BAC, \text{ or } ab = AB \,.\, \cos \text{inclination}.$$

4. The orthographic projection of a circle is an ellipse whose axis major is equal to the diameter of the circle, and axis minor = axis major × cosine of inclination of the plane of the circle to the primitive.

Let AB (fig. 77.) be that diameter which is parallel to the plane of projection, EF the diameter perpendicular to AB, and consequently to the line of intersection of the plane of the circle and the primitive; therefore the inclination of EF to the latter plane measures the inclination of the two planes; also let PN be any ordinate parallel to EC, and ab, ef, pn the projections of these lines. Then

$$pn = PN \,.\, \cos \text{incl}^{\text{n}} = \frac{ec}{EC} \,.\, PN = \frac{ec}{ac}\, qn,$$

aqb being a semicircle on the diameter ab; therefore the locus of p, or the projection of the circle, is an ellipse whose

$$\text{major axis } ab = AB, \text{ and minor axis } ef = ab \,.\, \cos \text{inclination}.$$

The orthographic projection of a circle parallel to the primitive is an equal circle; that of a circle perpendicular to the primitive a straight line equal to its diameter.

5. The stereographic projection of any circle not passing through the place of the eye is a circle.

Let E be the place of the eye (fig. 78.) at the extremity of the diameter NE, which is perpendicular to the primitive FG, AQB any small circle whose pole is P, $PNFE$ the great circle passing through E and P, and whose plane is consequently perpendicular to the primitive and to the plane of AQB, and bisects the circle AQB in AB. Let aqb be the intersection of the cone whose vertex is E and directrix AQB with the primitive, or the projection of AQB; and let ab be the line in which this projection is intersected by the plane AEB which is the principal section of the cone; then, joining NB, we have

$$\angle EAB = ENB = \text{compl}^{\text{t}}. \text{ of } BEN = CbE.$$

Hence (Conic Sections, Art. 221.) aqb is a subcontrary section of the cone and therefore a circle having ab for its diameter.

6. Having given the angular radius of a circle and the inclination of its plane to the primitive, to find the center and radius of its stereographic projection.

From any point Q in the given circle draw QE meeting the primitive in q; join CQ, Cq, and let NQ, PQ be arcs of great circles. Then from the spherical triangle NPQ, taking as usual the arcs to represent the angles which they subtend,

$$\cos PQ = \cos NP \cos NQ + \sin NP \sin NQ \cos PNQ.$$

Let $Cq = \rho$, $\angle\, aCq = \theta$, and the radius of the sphere $= r$;

$$\therefore \frac{\rho}{r} = \tan CEq = \tan \tfrac{1}{2} NQ\,; \;\therefore \sin NQ = \frac{2r\rho}{r^2+\rho^2},\; \cos NQ = \frac{r^2-\rho^2}{r^2+\rho^2},$$

also let the sines and cosines of the angles subtended by PQ, PN, be respectively denoted by s, c, s', c'; therefore by substitution,

$$c = c'\frac{r^2-\rho^2}{r^2+\rho^2} + s'\frac{2r\rho}{r^2+\rho^2}\cos\theta\,;$$

$$\therefore\; (c+c')\rho^2 - 2rs'\rho\cos\theta + (c-c')r^2 = 0,$$

$$\text{or }\; \rho^2 - 2r\frac{s'}{c+c'}\rho\cos\theta + \frac{c-c'}{c+c'}r^2 = 0,$$

which is the polar equation to the circle aqb. Let O be its center, since it must lie in ab, and let CO, the distance of the centers of the primitive and the projection, $= D$, and Oq, the radius of the projection, $= R$; $\therefore\; \rho^2 - 2D\rho\cos\theta + D^2 - R^2 = 0$; therefore, comparing this with the above equation,

$$D = \frac{rs'}{c+c'} = \frac{r\sin PCN}{\cos PCQ + \cos PCN};$$

$$D^2 - R^2 = \frac{c-c'}{c+c'}r^2;$$

$$\therefore\; R^2 = \frac{r^2 s'^2}{(c+c')^2} - \frac{c-c'}{c+c'}r^2 = r^2\,.\,\frac{s'^2 - c^2 + c'^2}{(c+c')^2} = \frac{r^2 s^2}{(c+c')^2},$$

$$\text{or }\; R = \frac{rs}{c+c'} = \frac{r\sin PCQ}{\cos PCQ + \cos PCN},$$

where PCQ, PCN, are respectively the angular radius, and inclination to the primitive, of the original circle.

The direction in which D is measured, coincides with the plane of the great circle passing through the eye and the pole of the circle to be projected. If the circle intersect the primitive, the line joining the points of intersection will be a chord of the projection, and the center of the projection will be in a line bisecting that chord at right angles.

7. The following particular cases are to be noticed.

(1) Let the original be a great circle, then $PCQ = 90^0$;

$$\therefore\ D = r\tan PCN = r\ .\ \text{tan inclination},$$

$$R = r\sec PCN = r\ .\ \text{sec inclination}.$$

(2) Let the original be parallel to the primitive, then $PCN = 0$;

$$\therefore\ D = 0, \quad R = \frac{r\sin PCQ}{1+\cos PCQ}$$

$$= r\tan\tfrac{1}{2}PCQ = r\tan\tfrac{1}{2}\,(\text{angular radius}).$$

(3) Let the original be perpendicular to the primitive, then $PCN = 90^0$;

$$\therefore\ D = r\sec PCQ = r\ .\ \text{sec angular radius},$$

$$R = r\tan PCQ = r\ .\ \text{tan angular radius}.$$

8. If the greatest and least angular distances NCA, NCB of the circumference of the original from N the opposite pole of the primitive be given equal to α and β,

$$\text{then } R = \tfrac{1}{2}(Ca + Cb)$$

$$= \frac{r}{2}(\tan\tfrac{1}{2}\alpha + \tan\tfrac{1}{2}\beta) = \frac{r\sin\frac{1}{2}(\alpha+\beta)}{2\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta},$$

$$D = R - Cb = \tfrac{1}{2}(Ca - Cb)$$

$$= \frac{r}{2}(\tan\tfrac{1}{2}\alpha - \tan\tfrac{1}{2}\beta) = \frac{r\sin\frac{1}{2}(\alpha-\beta)}{2\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}.$$

If β be greater than α, D becomes negative; which shews that O is situated to the left of C; that is, the center of the projection lies always between the center of the primitive, and the line of intersection of the planes of the original and primitive. If B fall to the right of N, the sign of β must be changed in the above expressions.

9. The projection of the arc NPA is Ca, which is equal to $CE \tan CEa$, or $CE \tan \frac{1}{2} NCA$; that is, the projection of an arc of a great circle, one of whose extremities coincides with the pole of the primitive, is equal to the radius multiplied by the tangent of half the angle which the arc subtends at the center. Every circle whose plane passes through the eye is of course projected into a straight line.

10. The angle between any two great circles is equal to the angle which their stereographic projections make with each other at the point in which they intersect.

Let two great circles AM, BM, intersect in the point M (fig. 85.), and let Mt, Mt', be lines touching them at that point; also let NME be the great circle passing through M, and the eye E. Let the tangent plane to the sphere at the point M, which will contain the lines Mt, Mt', cut the plane of NME in the line MT, and the primitive in the line Tt'; then if m be the projection of M, and mt, mt' be joined, $\angle tmt'$ is the projection of $\angle tMt'$. But Tt being the intersection of two planes, each of which is perpendicular to the plane NME, is itself perpendicular to the same plane, and therefore to the lines TM, Tm; and the angles TMm, TmM are equal, being the complements of equal angles CMm, CEM; $\therefore TM = Tm$, and the triangles Tmt, tTM, are equal in all respects; $\therefore \angle TMt = Tmt$; similarly $TMt' = Tmt'$; $\therefore tMt' = tmt'$, or $\angle AMB = \angle AmB$.

11. It is evident that tMt' is the angle between any two curves on the surface of the sphere intersecting in M and having the same tangent lines as the great circles, and tmt' is the angle between their projections.

12. *The gnomonic projection of every small circle is a conic section.*

Let DE (fig. 79.) be any small circle whose pole in N, AN a great circle passing through A, the center of the primitive, and through N, and consequently bisecting the circle DE, and meeting the primitive in the line AO. Then if the right cone whose vertex is P and directrix EQD intersect the primitive in the curve GqH, this will be the projection of EQD; which is therefore a conic section; and the projection will evidently be an ellipse when the original lies entirely above or below the plane through the eye parallel to the primitive; a parabola when it touches that plane; and a hyperbola when it intersects that plane.

13. Having given the angular radius of a circle and the inclination of its plane to the primitive, to find the center and axes of its gnomonic projection.

Constructing as before, take any point Q in the given circle and produce PQ to meet the primitive in q, also join AQ, NQ, by arcs of great circles, and let $An = x$, $qn = y$, be rectangular co-ordinates of q, and $PA = r$; then from the spherical triangle ANQ,

$$\cos NQ = \cos NA \cos QA + \sin NA \sin QA \cos NAQ;$$

$$\text{but } \tan AQ = \frac{Aq}{PA} = \frac{1}{r}\sqrt{x^2 + y^2},$$

$$\cos NAQ = \cos qAn = \frac{x}{\sqrt{x^2 + y^2}};$$

hence, denoting the sines and cosines of the angles subtended by NQ, NA, by s, s', c, c' respectively, and substituting,

$$c = c' \,.\, \frac{r}{\sqrt{x^2 + y^2 + r^2}} + s' \,.\, \frac{x}{\sqrt{x^2 + y^2 + r^2}};$$

$$\therefore y^2 = \frac{r^2}{c^2}(c'^2 - c^2) + 2\frac{s'c'}{c^2}rx - \frac{x^2}{c^2}(c^2 - s'^2) \ldots\ldots (1),$$

which is the equation to a conic section of which the axis of x is an axis; and if a and b be the semi-axes, and $d = AO$ the distance of the center of the projection GH from A, then

$$y^2 = \frac{b^2}{a^2}\{a^2 - (x - d)^2\} = \frac{b^2}{a^2}(a^2 - d^2 + 2dx - x^2);$$

therefore, comparing this equation with the above,

$$\frac{b^2}{a^2} = \frac{c^2 - s'^2}{c^2}, \quad \frac{b^2}{a^2}d = \frac{s'c'}{c^2}r, \quad \frac{b^2}{a^2}(a^2 - d^2) = \frac{r^2}{c^2}(c'^2 - c^2);$$

$$\therefore d = \frac{s'c'r}{c^2 - s'^2}, \quad a^2 = d^2 + r^2 \cdot \frac{c'^2 - c^2}{c^2 - s'^2} = \frac{r^2c^2s^2}{(c^2 - s'^2)^2},$$

$$\therefore a = \frac{rcs}{c^2 - s'^2}, \text{ and } b = \frac{rs}{\sqrt{c^2 - s'^2}},$$

in which expressions c, s, denote the sine and cosine of the angular radius, and c', s' those of the inclination.

14. Let α and β denote the greatest and least angular distances of the circumference of the original from A,

$$\text{then } c^2 - s'^2 = \cos^2 NPE - \sin^2 NPA = \cos (NPA + NPE).$$

$$\times \cos (NPA - NPE) = \cos\alpha \cos\beta;$$

$$\therefore d = \frac{r}{2}\frac{\sin 2\,NPA}{\cos\alpha\cos\beta} = \frac{r}{2}\frac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta},$$

$$a = \frac{r}{2}\frac{\sin 2\,NPE}{\cos\alpha\cos\beta} = \frac{r}{2} \cdot \frac{\sin(\alpha - \beta)}{\cos\alpha\cos\beta},$$

$$b = \frac{r \sin NPE}{\sqrt{\cos\alpha\cos\beta}} = r \cdot \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sqrt{\cos\alpha\cos\beta}}.$$

15. Let $DA = 90^\circ$, or $NQ + NA = 90^\circ$, and therefore $s' = c$; then equation (1) becomes

$$y^2 = \frac{r^2}{c^2}(s^2 - c^2) + 2\frac{s}{c}\,rx.$$

Let l = the latus rectum of the parabola represented by this equation, and d the distance of its vertex from A;

$$\therefore y^2 = l\,(x - d);$$

$$\therefore l = \frac{2sr}{c} = 2r \,.\, \tan \text{ (angular rad.)} \quad \text{and } ld = \frac{r^2}{c^2}(c^2 - s^2),$$

$$\text{or } d = \tfrac{1}{2}\frac{r}{cs}(c^2 - s^2) = r \cot 2 \text{ (angular rad.)}$$

16. If the given circle be parallel to the primitive, its projection will evidently be a circle, concentric with the pri-

mitive, whose radius = r tan (angular rad.) This also appears from Art. 13; for we have $s' = 0$, $c' = 1$;

$$\therefore a = b = r\frac{s}{c} = r \tan \text{(angular rad.)}$$

For the general method of solving problems of this kind see Hymers's Analytical Geometry of Three Dimensions, p. 183.

We shall now give an example of the application of the preceding propositions to drawing the projections of the parallels and meridians according to each of the three methods, preparatory to delineating a portion of the Earth's surface.

17. To delineate the orthographic projection of the circles of the terrestrial sphere upon the plane of the meridian of any place.

Let $WNES$ (fig. 88.) be the primitive, WE, NS being the projections of the equator and a meridian perpendicular to the primitive. Take $CB = r \cos L$, and describe the semi-ellipse NBS whose $\frac{1}{2}$ axes are CN, CB; then NBS is the projection of a meridian inclined at an angle L to the primitive; and similarly may all the meridians be projected. The parallels of latitude, since their planes are perpendicular to the primitive, will all be projected into straight lines parallel to WE. Having thus described the meridians and parallels, the countries intended to be represented are to be traced out according to their respective situations and extent, the position of every point being determined by the intersection of its given meridian and parallel of latitude. The distance, measured along CE, between two meridians in the projection whose originals are inclined to one another at a given angle δL, $= r \sin L \delta L$ nearly, and therefore decreases as the originals become less inclined to the primitive; hence the countries at a distance from the center of the primitive are contracted beyond their natural limits.

18. To delineate the gnomonic projection of the circles of the sphere upon a plane parallel to the equator.

In this case the meridians will all be projected into straight lines, being diameters of the primitive, and forming the same angles with one another that their originals do on the surface of the sphere; and the parallels will be projected into circles concentric with the primitive, the radius of any one being equal to r. tan (angular radius of its original). (Art. 16.) Hence the circumference of the primitive will be the projection of the parallel of 45^0; and for parallels of latitude less than 45^0 the projections fall without the primitive, and as the parallel approaches the equator, the radius of its projection increases, and at last becomes infinite. Consequently a whole hemisphere cannot be represented by this method; it affords a tolerably accurate representation of the polar regions, but all countries further from the pole than the parallel of 60^0 are distorted by too great extension. The gnomonic projection is the reverse of the orthographic as to its defects; both are best adapted for representing countries at a moderate distance from the center of the primitive.

19. To delineate the stereographic projection of the circles of the sphere upon the horizon of any place.

Let NS, WE, (fig. 88.) be the projections of the meridian and prime vertical, $NESW$ being the primitive. Then, because the projection of an arc of a great circle originating at the pole of the primitive $= r \tan$ ($\frac{1}{2}$ the angle which it subtends), (Art. 9.) taking on the projection of the meridian $CP = r \tan \frac{1}{2}$ colat., P is the projection of the pole, and taking $CQ = r.\tan \frac{1}{2}$ lat., the circle described through WQE is the projection of the equator. Let RPR' be the projection of a meridian inclined at an angle L, to the meridian of the place; the line joining R, R' is a diameter of the primitive, and O the center of RPR' lies in the line CO perpendicular to CR.

But $CO = r \tan$ inclination $= r \tan NRP = r \tan l \operatorname{cosec} NCR$ from the spherical triangle of which NPR is the projection,

$$\text{or } CO = r \tan l \sec ACO,$$

which shews that the centers of the projections of all the meridians lie in a straight line AO perpendicular to SC, where

$$AC = r \tan l.$$

Also, $\sin l = \tan NCR \cot NPR = \cot L \cot ACO$; if, therefore, we determine ACO from this equation, a circle described from center O with radius OP, is the projection of a meridian inclined at an angle L to the meridian of the place; hence the projections of all the meridians can be drawn. To project a parallel whose angular radius is δ, take Cr, Cs equal to $r \tan \frac{1}{2}(c-\delta)$, $r \tan \frac{1}{2}(c+\delta)$ respectively, c being the colatitude, then a circle described on the diameter sr, is the required projection.

20. All perspective representations of the sphere distort those parts which are not projected near the center of the primitive plane. In maps of small portions of the Earth, this is not of much consequence, as the middle of the map may be always taken for the center of the primitive, but when extensive tracts are to be represented, the countries near the edge of the map must differ considerably from their true proportion. Lagrange, in the Berlin Memoirs for 1779, has investigated modes of drawing charts which coincide most nearly with the countries to be represented; and other methods may be seen in the *Encyclopedie Methodique*, Art. Cartes. Here it is only necessary to notice Mercator's Projection, according to which sea-charts are usually drawn.

21. If we conceive the Earth to be touched according to its equator by a cylinder, and the meridians and parallels to be projected upon the surface of this cylinder, the eye being in the center, the meridians would manifestly be projected into straight lines parallel to the Earth's axis, and the parallels into circles whose planes are perpendicular to that axis; consequently when the cylindrical surface is developed, the meridians would be represented by equidistant parallel straight lines, and the parallels also by straight lines perpendicular to the meridians, the distance of any one from the projection of the equator being proportional to the tangent of its latitude. Hence for high latitudes, the distance of the parallels would become very great, and it would be impossible to represent the parts near the pole by this method; besides, on the sphere, if two places are on the same meridian, and their latitudes differ by 1^0, their absolute distance is every where the same, whereas in the projection we have feigned, it would vary nearly as the square of the secant of latitude; also if two places are on the same parallel on the sphere, and their longitudes differ by 1^0, their absolute distance varies as the cosine of latitude, but in the projection, it would be invariable. The above mode of projection would therefore be very defective, and the following

modification of it will give a much more accurate representation, viz. that the measures of a degree of latitude and longitude at any point of the projection, shall have the same relation as exists between their measures on the surface of the sphere at the corresponding point, the meridians being still projected upon the surface of the enveloping cylinder, and the parallels drawn consistently with the condition just stated. This is the principle of Mercator's method, which consequently does not give a *projection* as hitherto understood, that is, such a representation as would be formed by drawing lines from the eye, through every point of the object, to meet the surface on which it is to be delineated.

22. *To find the length of an arc of the meridian in Mercator's projection.*

Let s be the length of the arc of the meridian intercepted between the equator and a point whose latitude is l, and distance from the Earth's axis r; u the distance of the projection of the same point from the projection of the equator; h the circular measure of an angle of 1^0; then on the Earth's surface, at that point,

length of 1^0 of latitude $= d_l s \,.\, h$, nearly,

.... longitude $= r \,.\, h$;

and in the projection

length of 1^0 of latitude $= d_l u \,.\, h$, nearly,

.... longitude $= a \,.\, h$,

a being the radius of the equator; and the relation between these is to be the same, or, more strictly, the *limiting* ratio of the measures of equal increments of latitude and longitude is to be the same;

$$\therefore\ \frac{1}{a} d_l u = \frac{1}{r} d_l s.$$

If we suppose the Earth to be a sphere, $r = a \cos l$, $s = al$;

$$\therefore\ \frac{1}{a} d_l u = \sec l; \qquad \therefore\ \frac{u}{a} = \log_\varepsilon \left\{\tan\left(\frac{\pi}{4} + \tfrac{1}{2} l\right)\right\};$$

or, in a series,

$$u = a\left(l + \frac{1}{6}l^3 + \frac{1}{24}l^5 + \frac{61}{5040}l^7 + \&c.\right).$$

If we suppose the Earth a spheroid, (Arts. 125, 6.),

$$r = x = \frac{a\cos l}{(1 - e^2\sin^2 l)^{\frac{1}{2}}},\quad d_l s = \text{radius of curvature} = \frac{a(1 - e^2)}{(1 - e^2\sin^2 l)^{\frac{3}{2}}},$$

$$\therefore\ \frac{1}{a}\,d_l u = \frac{1 - e^2}{\cos l\,(1 - e^2\sin^2 l)} = \frac{1}{\cos l} - \frac{e^2\cos l}{1 - e^2\sin^2 l}$$

$$= \frac{1}{\cos l} - \frac{e}{2}\cdot\frac{2d_l(e\sin l)}{1 - e^2\sin^2 l};$$

$$\therefore\ \frac{1}{a}\,u = \log_\varepsilon\left\{\tan\left(45^0 + \tfrac{1}{2}l\right)\right\} - \frac{e}{2}\log_\varepsilon\left(\frac{1 + e\sin l}{1 - e\sin l}\right)$$

$$= \log_\varepsilon\left\{\tan\left(45^0 + \tfrac{1}{2}l\right)\right\} - e\left(e\sin l + \tfrac{1}{3}e^3\sin^3 l + \&c.\right)$$

$$= \frac{u_1}{a} - e^2\sin l \quad \text{nearly},$$

u_1 being the length, on the supposition of the Earth's being spherical.

23. Since the ratio of the measures of a degree of latitude and longitude is the same on the sphere and on the projection, any curve on the projection will cut the meridians at the same angles as its original cuts the meridians on the surface of the sphere. Hence the rhumb line, which on the sphere has the property of cutting all the meridians at the same angle, will, in Mercator's projection, be represented by a straight line.

APPENDIX II.

ON DIALLING.

1. IF a rectilinear gnomon be placed parallel to the Earth's axis, the angular motion of its shadow on a plane parallel to the equator will be uniform. For if the gnomon coincided with the Earth's axis, its shadow would revolve uniformly on any plane parallel to the equator, because the Sun's diurnal motion may be considered uniform during the same day; and since the distance between the gnomon and the Earth's axis subtends at the Sun an angle, the greatest value of which does not exceed $8'',57$, they may be considered coincident; therefore the shadow, cast by a gnomon placed parallel to the Earth's axis, will revolve uniformly on a plane parallel to the equator, at the rate of 15^0 an hour.

2. DEF. A line drawn from the center of the dial plane in the direction of the Earth's axis, is called the *Style*. A declination circle drawn through the style perpendicular to the dial plane intersects it in a line called the *Substyle*. The *Style's height* is the angle between the style and substyle. *Hour lines* are lines drawn from the center of the dial coincident with the shadow of the style, at the interval of each successive hour. The shadow at any instant is the intersection of the plane of the declination circle in which the Sun is at that instant, with the dial plane. Hence the twelve o'clock hour line, which coincides with the shadow of the style when the Sun is on the meridian, is the intersection of the plane of the meridian and the dial. Dials are denominated, according to the position of their planes, *Equatoreal*, *Horizontal*, &c.

3. *To construct an* Equatoreal Dial.

From the definitions, and Art. 1, it appears that in this dial the style is perpendicular to the dial plane, and the hour lines are drawn at intervals of 15^0, the 12 o'clock hour line being the intersection of the meridian with the dial plane. Since the dial is parallel to the equator, its inclination to the horizon = the co-latitude.

4. *To construct a* horizontal *Dial.*

Let $NHLS$ (fig. 80.) be the plane of the dial, OP the style, NO the 12 o'clock hour line, which, since the meridian passes through PO perpendicular to the horizon, is also the substyle; therefore PON, the style's height, = the altitude of the pole, or latitude of the place. At n hours after noon, let OH, to the east of ON, be the hour line; then, drawing the declination circle POH through OH, its plane will pass through the Sun, and therefore $\angle NPH = n.15^0$; therefore from the spherical triangle NPH in which $\angle PNH = 90^0$,

$$\sin PN = \tan NH \,.\, \cot n.15^0,$$

$$\text{or } \tan NOH = \sin \text{lat.} \tan n.15^0,$$

which determines the hour line OH; therefore giving n the values 1, 2, 3, &c. the hour lines corresponding to one, two, three o'clock, &c. may be drawn; and drawing similarly situated lines on the west side of ON, these are the hour lines corresponding to 11 o'clock, 10 o'clock, &c.

If OL be the last hour line on any day, l in LO produced is the Sun's place at setting, and Nl = the Sun's north azimuth at setting; but $\cos Pl = \cos PN \cos Nl$;

$$\therefore \cos Nl = \frac{\sin \text{Sun's dec.}}{\cos \text{lat.}};$$

therefore the greatest value of NL, which corresponds to the least value of Nl, or the extreme hour line on the dial, is found by the equation

$$-\cos NL = \frac{\sin 23^0.28'}{\cos \text{lat.}}.$$

5. *To construct a* vertical south *dial.*

Let ZLN (fig. 81.) be the dial plane coinciding with the prime vertical, Op the style directed towards the south pole, ZpN the meridian; therefore ON is the 12 o'clock hour line, and also, (since the meridian is perpendicular to the prime vertical) the substyle; and $NOp = ZOP$ = the co-latitude. Let OH be the hour line n hours from noon; then, as before,

$$\sin Np = \tan NH \,.\, \cot NpH;$$

$$\text{therefore } \tan NH = \cos \text{lat.} \tan n.15^0,$$

from which, by giving n the different values 1, 2, 3, &c. the different hour lines may be drawn, observing that the hour lines before noon are to the west of ON, and the others to the east.

If OL be the last hour line on any day, the Sun is then in the plane of the dial at l, and pL = the Sun's north polar distance $= 90^0 - \delta$; but

$$\cos pL = \cos Np \, . \cos NL \, ; \quad \therefore \; \sin \delta = \sin \text{lat.} \cos NL,$$

which equation determines NL. When $\delta = 0$, $NL = 90^0$, and OL is the six o'clock hour line at the equinox; it is useless to graduate the dial beyond this, because when the declination is south, the Sun sinks below the horizon before he is found in the declination circle perpendicular to the meridian, i. e. before six o'clock; and when the declination is north, he is north of the prime vertical, and therefore ceases to shine on the dial, before six o'clock.

6. *To construct a* vertical declining *dial inclined at a given angle to the meridian.*

Let ZSN (fig. 81.) be the dial plane passing through the vertical ZN, and inclined at a given angle pNH (e) to the meridian PZp; therefore ON is the 12 o'clock hour line. Let Op be the style, OS the substyle, which falls amongst the morning or evening hour lines, according as the dial fronts the east, or fronts the west as in the figure. Then, since Np = the co-latitude, we have, in the right-angled triangle pSN,

$\sin pS = \cos \text{lat.} \, \sin e$, which gives pS, the style's height,

$\cos e = \tan \text{lat.} \tan SN$, ... SN, the place of the substyle,

$\sin \text{lat.} = \cot e \, . \cot NpS$, ... NpS (ϕ).

Let now OH be the hour line n hours from 12 o'clock;

$$\therefore \; SpH = \phi \mp n \, . \, 15^0,$$

$-$ or $+$ according as the substyle and hour line are on the same or different sides of ON. Then in the right-angled triangle SpH, we have

$$\tan SH = \sin pS \, . \tan (\phi \mp n \, . \, 15^0),$$

which determines SH, and giving n the values, 1, 2, 3, &c. all the hour lines may be determined.

If OL be the last hour line on any day, the Sun is then in the plane of the dial, and we have given $pL = 90^0 -$ the Sun's declination, $pN =$ the co-latitude, and $pNL = e$, to find NL.

7. *To construct an* East *or* West *dial.*

Let POp (fig. 82.) be the dial which coincides with the plane of the meridian, O its center, Pp parallel to the Earth's axis, ASB the style which is parallel to Op, and formed by the edge of a rectangular lamina, AD, passing through Pp, perpendicular to the dial plane; then, since the plane AD is perpendicular to the meridian, CD is the six o'clock hour line; and since AB is parallel to the plane of the dial, all the other hour lines are parallel to AB or Pp. Let aHb be the hour line n hours from six o'clock, OSH a plane perpendicular to Pp or AB, intersecting the dial plane in the line OH; then, since the Sun revolves uniformly round AB, $OSH = n \cdot 15^0$, and SOH is a right angle;

$$\therefore \; OH = OS \cdot \tan n \cdot 15^0,$$

which determines the hour line ab, and giving different values to n, all the hour lines may be drawn.

As the Sun only shines on the dial during half the day, if the dial fronts the east, it points out the time from sun-rise to noon, or if the dial fronts the west, from noon to night.

8. *To construct a* general dial, *inclined at any given angles to the meridian and horizon.*

Let SON (fig. 83.) be the plane of the dial inclined at an angle HNR (N) to the meridian PZN, and at an angle RAN, (A) to the horizon; therefore, ON is the 12 o'clock hour line. Let OP be the style passing through the pole P, OS the substyle; therefore, $PSN = 90^0$. But $ARN = 90^0$; therefore, by Napier's rules,

$$\cos A = \sin N \cdot \cos RN,$$

whence RN, and, therefore, $PN =$ the latitude $+ RN$, are known. And from the right-angled triangle PSN,

$\cos PN = \cot SPN \cdot \cot N$, which gives SPN (ϕ),

$\cos N = \tan SN \cdot \cot PN$............ SN, the position of the substyle,

$\sin SP = \sin N \cdot \sin PN$...............$SP$, the style's height ($h$).

At n hours from noon, let OH be the hour line; therefore,

$$SPH = \phi - n \,.\, 15^0;$$

$$\therefore\ \tan SH = \sin h \,.\, \tan(\phi - n \,.\, 15^0),$$

from which equation, by giving n the values 1, 2, 3, &c. all the hour lines may be drawn.

9. The time pointed out by a dial is apparent time, but, in consequence of the atmospheric refraction, which elevates the Sun in a vertical circle, the shadow only points out this time exactly at noon, the dial being fast all the morning, and slow in the evening.

The gnomon is generally formed by a triangular lamina of metal, the sides of which are, the style, substyle, and a perpendicular let fall from one on the other. If the thickness of this lamina is considerable, the shadow of the western edge points out the morning hours, and the eastern the evening, and the hour lines must be drawn for each separately, so as to form two half dials with different centers.

A solar dial may be made use of for roughly determining the time by means of the Moon's shadow, if we know the time when the Moon is on the meridian. For the time pointed out by the shadow is the apparent time from the Moon's being on the meridian, nearly; therefore, adding the time on the dial to the time of the Moon's transit, the solar time is determined. The method is not accurate, as the Moon moves through a little more than 15^0 in a solar hour.

APPENDIX III.

ON INTERPOLATION.

1. It is often requisite in Astronomy, to determine the *general* relation between two quantities, from a number of corresponding values; as for instance, from a number of zenith distances taken at given intervals, to determine the zenith distance for any intermediate time. This problem, which is the same as that of finding the equation to a curve passing through a number of given points, is plainly indeterminate; but if the corresponding values are numerous and near to each other, the general relation, within the limits of the given quantities, may be found to a great degree of accuracy.

2. Let a, b, c, &c. be values of y corresponding to α, β, γ, &c. values of x, and let

$$y = Aa + Bb + Cc + \&c.,$$

where A, B, C, &c. are functions of x to be determined. The second member of this equation ought to reduce itself to a, when $x = \alpha$, which will be the case if all the coefficients but A vanish, when $x = \alpha$, and A become unity; but, in order that B, C, &c. may vanish when $x = \alpha$, they must contain the factor $(x - \alpha)$; and as the same reasoning applies to B and C, all but B must contain $(x - \beta)$, and all but C, $(x - \gamma)$.

Let, therefore, $A = A' . (x - \beta) . (x - \gamma) \dots$ which ought to equal 1, when $x = \alpha$;

$$\therefore\ A' . (\alpha - \beta) . (\alpha - \gamma) \dots = 1;\quad \therefore\ A' = \frac{1}{(\alpha - \beta) . (\alpha - \gamma) \dots};$$

$$\therefore\ A = \frac{(x - \beta) . (x - \gamma) \dots}{(\alpha - \beta) . (\alpha - \gamma) \dots};\ \text{similarly for } B,\ C,\ \&c.$$

$$\therefore\ y = \frac{(x - \beta) . (x - \gamma) \dots}{(\alpha - \beta) . (\alpha - \gamma) \dots}\, a + \frac{(x - \alpha) . (x - \gamma) \dots}{(\beta - \alpha) . (\beta - \gamma) \dots}\, b + \&c.$$

3. When the values of x are taken at equal intervals, as generally happens in Astronomy, a more simple formula will suffice.

Let a, b, c, &c. be values of y corresponding to the values of $x = 0, 1, 2, 3$, &c. Also let the results of subtracting each value of y from the succeeding one be d_1, d'_1, d''_1, &c.; and the results of subtracting each of these differences from the succeeding one d_2, d'_2, d''_2, &c.; and performing the same operation on these new differences, let the results be d_3, d'_3, d''_3, &c.

$$\text{Let } y = a + Bx + C \, . \, x \, . \frac{x-1}{2} + D \, . \, x \, . \frac{x-1}{2} \, . \frac{x-2}{3} + \text{\&c.}$$

(B, C, &c. being constants); then making $x = 0, 1, 2$, &c. successively, we get

$$a = a,$$

$$b = a + B; \quad \therefore B = b - a = d_1, \text{ and } c - b = d'_1;$$

$$c = a + 2B + C;$$

$$\therefore C = c - 2b + a = (c - b) - (b - a) = d_2, \text{ and } d - 2c + b = d'_2;$$

$$d = a + 3B + 3C + D;$$

$$\therefore D = d - 3c + 3b - a = (d - 2c + b) - (c - 2b + a) = d_3;$$

$$\text{\&c.} = \text{\&c.}$$

$$y = a + d_1 \, . \, x + d_2 \, . \, x \, . \frac{x-1}{2} + d_3 \, . \, x \, . \frac{x-1}{2} \, . \frac{x-2}{3} + \text{\&c.}$$

a formula which gives exactly the values of y corresponding to $x = 0, 1, 2$, &c.; and approximately, the values of y corresponding to all intermediate values of x.

In general, the quantities d_1, d_2, &c. diminish very fast, and it will not often be necessary to proceed farther than d_3.

4. The following example will shew some of the different uses that may be made of the formulæ of interpolation.

Ex. 1. Given the Sun's declination at 12 o'clock on the 19th, 20th, 21st, 22d of June,

$$= 23^0 . 26' . 39'', \quad 23^0 . 27' . 23'', \quad 23^0 . 27' . 42'', \quad 23^0 . 27' . 37'',$$

respectively; to find the declination on the 20th at 6 o'clock.

$$\begin{array}{rccccccc}
a,\ b,\ c,\ d = & 23^0.26'.39''; & & 23^0.27'.23''; & & 23^0.27'.42''; & & 23^0.27'.37'', \\
d_1,\ d_1',\ d_1'' = & & 44'' & & 19'' & & -5'', & \\
d_2,\ d_2' = & & & -25'' & & -24'', & & \\
d_3 = & & & & 1'', & & &
\end{array}$$

and $x = 1\frac{1}{4} = \frac{5}{4}$,

therefore, the required declination

$$= 23^0.26'.39'' + \frac{5}{4}.44'' - \frac{5}{4}.\frac{1}{4}.\frac{25''}{2} - \frac{5}{4}.\frac{1}{4}.\frac{3}{4}.\frac{1''}{2.3} + \&c.$$

$$= 23^0.27'.37''.$$

The declination (y) at any time x

$$= 23^0.26'.39'' + x.44'' - x.\frac{x-1}{2}.25''.$$

Since the declination increases and then diminishes, it admits at some intermediate time of a maximum, the time being determined by the equation

$$44'' - \frac{2x-1}{2}.25'' = 0;$$

$$\therefore\ x = \frac{113}{50} = \frac{226}{100} = 2.26^{\text{d}} = 2^{\text{d}}.6^{\text{h}}.14'.24'',$$

and substituting $\frac{113}{50}$ for x, $y = 23^0.27'.42'',8$.

Therefore the Sun's greatest declination (or the obliquity) happens on June 21, at $6^{\text{h}}.14^{\text{m}}.24^{\text{s}}$, and $= 23^0.27'.42''$.

Ex. 2. To find the Greenwich mean time at which the distance between the Moon and α Arietis was $26^0.6'.12''$, having given that at 6^{h}, 9^{h}, and midnight, its value was, respectively,

$$25^0.26'.43'', \quad 26^0.47'.56'', \quad 28^0.9'.50''.$$

These give $d_1 = 1^0.21'.13''$, $d_2 = 41''$;

$$\therefore 26^0.6'.12'' = 25^0.26'.43'' + (1^0.21'.13'')\,x - 41''.\frac{x(1-x)}{2};$$

hence, for a first approximation, $x = \frac{2369}{4873} = ,486$;

$$\therefore 2369 = x\,(4873 - 41 \times ,257)\,;$$

$$\therefore x = ,4872\,;$$

$\therefore$ the time required $= 6^h + (,4872)\,.\,3^h = 7^h.\,27^m.\,41^s.$

For a fuller account of the method of Interpolation, see a Treatise on Differential Equations and the Calculus of Finite Differences, p. 105.

5. In astronomical investigations, it is sometimes required to determine the values of several unknown quantities from linear equations, which are more in number than the quantities to be determined. The coefficients of these equations are generally determined by observation, and if the observations are correct, the values of the unknown quantities deduced from some of the equations ought, when substituted, to satisfy the remainder; but as this is never the case, the unknown quantities ought to be so determined, as to be least subject to error. If the equations be combined, so that the factor of one of the unknown quantities, as x, is the greatest possible, and the coefficients of the others, y, z, &c. the least possible, x will be determined most accurately, since any errors in the values of y, z, &c. will have less influence on the value of x, being multiplied by the least factors, and divided by the greatest coefficient of x. In the same manner equations may be formed, to give the values of y, z, &c. most accurately, and from all these equations, which will be the same in number as the unknown quantities, x, y, z, &c. may be found. The equations for determining x, y, &c. may be formed by changing the signs, so that all the coefficients of x may have the same sign, and then adding, and proceeding in the same way with y, z, &c.

Ex. Let the equations be

$$3 - x + y - 2z = 0,$$

$$5 - 3x - 3y + 5z = 0,$$

$$21 - 4x - y - 4z = 0,$$

$$14 + x - 3y - 3z = 0,$$

changing the sign of the last equation, and adding

$$15 - 9x + y + 2z = 0,$$

similarly for y, $37 - 5x - 7y = 0$,

for z, $33 - x - y - 14z = 0$.

From these equations, $x = 2.486$,

$$y = 3.517,$$

$$z = 1.929,$$

which three values substituted in the original equations give, instead of 0, the results

$$0.175, \quad 0.141, \quad -0.173, \quad 0.151.$$

Second Method.

$$\begin{aligned} \text{Let } m + ax + by + cz + \&c. &= 0, \\ m' + a'x + b'y + c'z + \&c. &= 0, \\ m'' + a''x + b''y + c''z + \&c. &= 0, \\ \&c.\ldots\ldots\ldots\ldots &= 0, \end{aligned}$$

be the equations; multiply the first by a, the second by a', and so on; then, by addition,

$$ma + m'a' + \&c. + (a^2 + a'^2 + \&c.)x + (ab + a'b' + \&c.)y + (ac + a'c' + \&c.)z + \&c. = 0.$$

Similarly,

$$(mb + m'b' + \&c.) + (ab + a'b' + \&c.)x + (b^2 + b'^2 + \&c.)y + (bc + b'c' + \&c.)z + \&c. = 0,$$

$$(mc + m'c' + \&c.) + (ac + a'c' + \&c.)x + (bc + b'c' + \&c.)y + (c^2 + c'^2 + \&c.)z + \&c. = 0,$$

$$\&c. + \&c.\ldots\ldots\ldots\ldots\ldots\ldots = 0.$$

By this means as many equations are formed as there are unknown quantities, and from them x, y, z, &c. may be determined.

The method applied to the example in the preceding article, gives the reduced equations

$$-88 + 27x + 6y = 0,$$

$$-70 + 6x + 15z = 0,$$

$$-107 + y + 54z = 0.$$

From whence $x = 2.470$; $y = 3.551$; $z = 1.916$

The above method of reducing the linear equations which is called the *Method of Least Squares*, was invented by Gauss, and is proved by him in his *Theoria motus Corporum Cœlestium*, to give the most probable values of the unknown quantities. The proof is not sufficiently elementary to be given here. See also Laplace, *Theorie des Probabilities*, Chap. IV. and the *Berlin Ephemeris* for 1834.

APPENDIX IV.

PROBLEMS.

THE following Problems are added here, partly with a view of assisting the student in acquiring a familiarity with the Definitions, and a readiness in applying Spherical Trigonometry to this subject; and partly because considerable interest attaches to some of them in the History of Astronomy.

I. *To find the time from observing when two known stars have the same altitude, the altitude itself being unknown.*

Let a, a' be the right ascensions of the two stars, δ, δ' their declinations, h, h' the hour angles, z the common zenith distance, and l the latitude of the place.

Then, if the stars are observed at the same instant,

$$h' - h = a' \sim a\,;$$

if they are observed after an interval of t sidereal hours,

$$h' - h = a' \sim a \pm 15^0 t \text{ (Art. 158.)}.$$

Let this quantity $= \alpha$; $\therefore h' = h + \alpha$, and we have

$$\cos z = \cos l \,.\, \cos \delta \,.\, \cos h + \sin l \,.\, \sin \delta,$$

$$\cos z = \cos l \,.\, \cos \delta' \,.\, \cos (h + \alpha) + \sin l \,.\, \sin \delta'$$

$$= \cos l \,.\, \cos \delta' \,.\, \cos \alpha \,.\, \cos h - \cos l \,.\, \cos \delta' \sin \alpha \sin h + \sin l \,.\, \sin \delta'\,;$$

therefore, subtracting,

$$0 = \cos l \cos h \, (\cos\delta' \cos a - \cos\delta)$$
$$- \cos l \, . \cos\delta' \sin a \sin h + \sin l \, . \, (\sin\delta' - \sin\delta) \text{ ; or}$$
$$(\cos\delta' \cos a - \cos\delta) \, . \cos h - \cos\delta' \, . \sin a \, . \sin h = \tan l \, . (\sin\delta - \sin\delta').$$

Let ϕ be determined from the expression

$$\cot\phi = \frac{\cos\delta' \, . \cos a - \cos\delta}{\cos\delta' \sin a} \, ;$$

$$\therefore \cos\delta' \, . \sin a \, . \, (\cot\phi \cos h - \sin h) = \tan l \, . \, (\sin\delta - \sin\delta') \, ;$$

$$\therefore \cos\delta' \, . \sin a \, . \frac{\cos(\phi + h)}{\sin\phi} = 2 \tan l \, . \sin\tfrac{1}{2}(\delta - \delta') \, . \cos\tfrac{1}{2}(\delta + \delta') \, ;$$

$$\therefore \cos(\phi + h) = \frac{2 \sin\phi \, . \tan l}{\cos\delta' \, . \sin a} \, . \sin\tfrac{1}{2}(\delta - \delta') \, . \cos\tfrac{1}{2}(\delta + \delta'),$$

from which h, and therefore the time may be obtained.

II. When the latitude and time are determined from two altitudes and the time between, to find the errors caused by given small errors in the observed altitudes.

Let P (fig. 19.) be the pole, Z the zenith, and S the Sun at the first observation; Z' and S' the places of Z and S determined from the erroneous altitudes; then since the Sun's declination is supposed to be known without error, $PS = PS'$, and SS' is a parallel of declination. Let δl, δz, $\delta z'$, δh, $\delta h'$, be the cotemporary errors of l, z, z', h and h', where, since the interval $h' - h$ between the observations is supposed to be known correctly, $\delta h' = \delta h$; then drawing Zm, Sn, perpendiculars to $Z'S'$ and making $ZZ'S'$ the south azimuth of $S = a$, we have

$$\delta z = Z'S' - ZS = Z'm + S'n,$$
$$= \delta l \, . \cos a + SS' \, . \cos SS'n,$$

but $SS' \, . \cos SS'n = \delta h \sin PS \, . \sin ZSP$

$$= \delta h \, . \sin PZ \, . \sin PZS$$
$$= \delta h \, . \cos l \, . \sin a \, ;$$
$$\therefore \delta z = \delta l \cos a + \delta h \, . \cos l \, . \sin a.$$

Similarly, if a' be the azimuth at the second observation,

$$\delta z' = \delta l \cos a' + \delta h \, . \cos l \sin a' ;$$

$$\therefore \delta l = \frac{\delta z \sin a' - \delta z' \sin a}{\sin (a' - a)} ;$$

$$\delta h = \frac{\delta z' \cos a - \delta z \cos a'}{\cos l \, . \sin (a' - a)} .$$

Hence, in determining the latitude and time from two altitudes and the time between, $a' - a$ ought to be as near 90^0 as possible; so that if the one observation is made near the prime vertical, the other must be made near the meridian.

III. To find the Sun's longitude, or the day of the year, when a given star rises heliacally.

When the Sun in his passage through the ecliptic is a little to the westward of a star, the star sets a little after him; and it is said to set *heliacally* when the Sun is just so far separated from it, as, by being sufficiently below the horizon, to admit of the star's being seen at setting. After the interval of a day or two, the Sun by his increased proximity will overpower the star's light at setting, and will soon get to the eastward of it; and the setting of the star taking place in broad day light, will of course be invisible. The star will then rise before the Sun, and will soon precede him just enough to be visible in the horizon at the dawn; the first time this happens is the day of its heliacal rising. For the same star the phenomena of heliacal rising and setting recur every year with the same positions of the Sun in the ecliptic; hence in ancient times, and in default of better means, they were employed to fix epochs. The necessary depression of the Sun below the horizon depends on the magnitude of the star, the clearness of the atmosphere, and other circumstances, but in general for stars of the first magnitude, the distance of the Sun below the horizon lies between 10^0 and 12^0.

Let ΥS (fig. 21. Pate II.) be the ecliptic, ΥQ the equator, Rs the horizon of the place of observation; therefore $\angle \Upsilon Qs =$ the colatitude; let s be a star rising heliacally, ΥT, Ts, its right ascension and declination, and S the Sun 10^0 below the horizon. Then, in the triangle TsQ,

$$\sin TQ = \tan \text{latitude} \times \tan \text{star's declination},$$

which determines TQ; $\therefore\ \Upsilon Q =$ the star's right ascension $\pm TQ$, is known.

Hence, in triangle ΥQO, we know

$$\Upsilon QO = 90^0 + l,\ Q\Upsilon O = \omega, \text{ and } \Upsilon Q;$$

therefore ΥO and QO may be found from the equations

$$\tan \tfrac{1}{2}(\Upsilon O + QO) = \frac{\cos \frac{1}{2}(Q - \Upsilon)}{\cos \frac{1}{2}(Q + \Upsilon)} . \tan \tfrac{1}{2} \Upsilon Q,$$

$$\tan \tfrac{1}{2}(\Upsilon O - QO) = \frac{\sin \frac{1}{2}(Q - \Upsilon)}{\sin \frac{1}{2}(Q + \Upsilon)} . \tan \tfrac{1}{2} \Upsilon Q.$$

Hence, $\sin O = \sin Q . \dfrac{\sin \Upsilon Q}{\sin \Upsilon O}$ is known; but from the right-angled triangle SOR, $\sin 10^0 = \sin SO . \sin O$, which determines SO; and hence we know $\Upsilon S = \Upsilon O + OS =$ the Sun's longitude, from whence, as has been shewn, the time of the year may be found.

A star rises cosmically which rises at the same time with the Sun. If s be a star rising *cosmically*, O is the place of the Sun; and ΥO, the Sun's longitude, may be found as before.

IV. To find when the Sun's azimuth increases fastest.

Denoting the Sun's declination, azimuth, and hour angle, by δ, A and h, and the latitude of the place of observation by l, we have

$$\cot A = \frac{\tan \delta \cos l - \cos h \sin l}{\sin h};$$

$$\therefore\ (\operatorname{cosec} A)^2 d_h A = \tan \delta \cos l \operatorname{cosec} h \cot h - \sin l (\operatorname{cosec} h)^2;$$

$$\therefore\ d_h A = \left(\frac{\sin A}{\sin h}\right)^2 (\tan \delta \cos l \cos h - \sin l)$$

$$= \left(\frac{\cos \delta}{\cos a}\right)^2 \left\{\frac{\sin \delta}{(\cos \delta)^2} (\sin a - \sin l \sin \delta) - \sin l\right\}$$

$$= \frac{1}{(\cos a)^2} \{\sin \delta \sin a - \sin l\} = \text{a maximum};$$

$$\therefore\ (\sin a)^2 - 2 \frac{\sin l}{\sin \delta} \sin a + 1 = 0,$$

$$\text{or } \sin a = \frac{\sin l}{\sin \delta} \pm \sqrt{\left(\frac{\sin l}{\sin \delta}\right)^2 - 1}.$$

Let ϕ be the star's altitude as it crosses the prime vertical; $\therefore \sin \delta = \sin l \sin \phi$;

$$\therefore \sin a = \frac{1 \pm \cos \phi}{\sin \phi} = \cot \frac{\phi}{2} \text{ or } \tan \frac{\phi}{2};$$

the value $\tan \frac{\phi}{2}$ must be taken, for ϕ is less than 90^0.

V. *To find the time of year when the twilight is shortest.*

Let Z (fig. 27. Plate III.) be the zenith, P the pole, ZQM a parallel to the equator, S the Sun at the commencement of twilight; therefore $ZS = 108^0$. Let the great circle which has S for its pole, cut ZQM in Z'; then ZPS is the hour angle when twilight begins; and since PZ' is the co-latitude, PS the north polar distance of the Sun, and $Z'S = 90^0$, $Z'PS$ is the hour angle at sun-rise; therefore the difference of these angles ZPZ' measures the duration of twilight, and is least when ZZ' the arc of a great circle joining Z and Z', is least; but ZZ' is never less than $SZ - SZ'$ or 18^0, and is least when it exactly equals $SZ - SZ'$, in which case Z' coincides with Q, and $ZQ = SZ - SZ' = 18^0$. Hence it appears that if on a vertical circle we take $ZS = 108^0$, and bring it into such a position that the intercepted arc ZQ is 18^0, then S is the position of the Sun at the beginning of twilight on the day when its duration is a minimum, and SP is the required north polar distance of the Sun. Draw PO perpendicular to SZ, and bisecting ZQ and $\angle ZPQ$, and let l be the latitude of the place, δ the declination of the Sun, and $\angle ZPQ = 2h$. Then, by Napier's Rules,

$$\cos ZP = \cos PO \,.\, \cos ZO, \text{ or } \sin l = \cos PO \,.\, \cos 9^0,$$

$$\cos PS = \cos PO \,.\, \cos SO, \text{ or } \sin \delta = -\cos PO \,.\, \sin 9^0;$$

$$\therefore \frac{\sin \delta}{\sin l} = -\tan 9^0,$$

$$\text{and } \sin \delta = -\tan 9^0 \,.\, \sin l \ldots\ldots (1).$$

Again, $\sin ZO = \sin ZP \,.\, \sin ZPO$,

$$\text{or } \sin 9^0 = \cos l \,.\, \sin h;$$

$$\therefore \sin h = \sin 9^{0} . \sec l \, \ldots\ldots\ldots (2).$$

The equation (1) gives the Sun's declination, or the time of year when the twilight is shortest, and equation (2) gives the duration of it.

It is manifest that PQS is equal to the Sun's azimuth at the end of twilight, and $PZQ = PQZ$ is the Sun's azimuth at the beginning, hence on the day of shortest twilight, the Sun's azimuths at the beginning and end of twilight are supplements to each other.

If it were required more generally to find at what time of year the Sun passes from one zenith distance z' to another z in the shortest time, we should similarly find that ZS must equal z' and $SQ = z$, and then from the triangles ZOP, SOP

$$\sin \delta = \sin l . \frac{\cos \frac{z' + z}{2}}{\cos \frac{z' - z}{2}}, \quad \sin h = \sin \frac{z' - z}{2} \sec l.$$

VI. *To find the equation to the curve which a star appears to describe about its true place in consequence of aberration, upon any hypothesis of the form of the Earth's orbit.*

From Art. 287 it appears that the apparent place of a star describes about its true place, in a plane parallel to the ecliptic, a curve of such a nature that a radius vector being drawn in the direction of the Earth's motion at any time, its length is proportional to the corresponding velocity of the Earth. Let PT fig. 36, be a tangent to the orbit described by the Earth about the Sun S, $r = SP$, $p =$ perpendicular on PT from S; take any point s in the line AS to represent the true place of a star, and draw $s\sigma$ parallel to PT and proportional to the velocity at P, and therefore $= \frac{c^2}{p}$; then the locus of σ is similar to the curve described by the apparent place about the true place. Let $\rho = s\sigma$, $\pi =$ perpendicular from s on the tangent at σ, $ASP = \theta$, $As\sigma = \phi$; then, since $s\sigma$ is parallel to PT,

$$\sin (\phi - \theta) = \sin SPT = \frac{p}{r};$$

therefore, taking the differential coefficients of the logarithms of both sides,

$$\cot(\phi-\theta)\,.\,\{d_\theta\phi-1\}=\frac{1}{p}\,d_\theta p-\frac{1}{r}\,d_\theta r\,;\ \text{but}\ \cot(\phi-\theta)=\frac{1}{r}\,d_\theta r,$$

$$\therefore\ \cot(\phi-\theta)\,d_\theta\phi=\frac{1}{p}\,d_\theta p,\ \text{or}\ \cot(\phi-\theta)=\frac{1}{p}\,d_\phi p=-\frac{1}{\rho}\,d_\phi\rho,$$

$$\text{since}\ \rho=\frac{c^2}{p}\,;\ \therefore\ \frac{1}{r}\,d_\theta r=-\frac{1}{\rho}\,d_\phi\rho,$$

$$\text{or}\ \frac{\sqrt{r^2-p^2}}{p}=-\frac{\sqrt{\rho^2-\pi^2}}{\pi}\,;$$

$$\therefore\ \frac{r^2}{p^2}=\frac{\rho^2}{\pi^2},\ \text{or}\ r\pi=p\rho=c^2\,;$$

$$\therefore\ r=\frac{c^2}{\pi},\ \text{and}\ p=\frac{c^2}{\rho}\,;$$

if therefore $r=f(p)$ be the equation to the orbit of P, the equation to the aberratic curve is

$$\frac{c^2}{\pi}=f\left(\frac{c^2}{\rho}\right).$$

Ex. Suppose the Earth's orbit an ellipse, of which the semi-axes are a and b; then, the equation between r and p is

$$p^2=\frac{b^2r}{2a-r},\quad\text{or}\quad r=\frac{2ap^2}{b^2+p^2}\,;$$

$$\therefore\ \pi=\frac{c^2}{r}=\frac{c^2}{2a}\cdot\frac{b^2+p^2}{p^2}$$

$$=\frac{c^2}{2a}\cdot\frac{b^2\rho^2+c^4}{c^4},\ \text{since}\ p=\frac{c^2}{\rho},$$

$$=\frac{b^2\rho^2+c^4}{2ac^2}.$$

We may easily shew that this is the equation to a circle; for let S (fig. 37.) be a point in a circle rad. CP (m), at a distance SC (n) from the center, and taking any point P in the circumference, let $SP=\rho$, $SY=\pi$,

$$\therefore\ n^2 = \rho^2 + m^2 - 2m\rho\,.\cos SPC$$

$$= \rho^2 + m^2 - 2m\pi\,;$$

$$\therefore\ \pi = \frac{m^2 - n^2 + \rho^2}{2m},$$

which coincides with the equation to the aberratic curve, if

$$m = a\,.\frac{c^2}{b^2}, \quad \text{and}\ m^2 - n^2 = \frac{c^4}{b^2}.$$

Hence it appears that the stars describe circles in consequence of aberration, though no longer having the true places in their centers, even if the Earth's elliptic motion be taken into consideration. As in Art. 288, unless the star be situated in the pole of the ecliptic, the circle, its plane being viewed obliquely, will appear an ellipse.

VII. *To find when the equation of the center is a maximum, and its greatest value.*

$$m = u - e\sin u, \quad \tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2};$$

$$\therefore\ d_u m = 1 - e\cos u, \quad d_u v = \frac{\sqrt{1-e^2}}{1 - e\cos u}.$$

But since $v - m$ is a maximum, $d_u(v - m) = 0$, or $d_u v = d_u m$;

$$\therefore\ 1 - e\cos u = \sqrt[4]{1 - e^2} = \sqrt{\cos\epsilon}, \ \text{if}\ e = \sin\epsilon.$$

$$\text{Hence}\ \frac{1 - e^2}{1 + e\cos v} = \sqrt[4]{1 - e^2}, \ \text{or}\ \cos v = \frac{(1 - e^2)^{\frac{3}{4}} - 1}{e}.$$

$$\text{Also}\ \tan\frac{v}{2} - \tan\frac{u}{2} = \frac{\sqrt{1+e} - \sqrt{1-e}}{\sqrt{1-e}}\tan\frac{u}{2};$$

$$\therefore\ \sin\frac{v-u}{2} = \frac{2\sin\frac{1}{2}\epsilon}{\sqrt{1-e}}\sin\frac{u}{2}\cos\frac{v}{2}.$$

$$\text{But}\ \cos v = \frac{\cos u - e}{1 - e\cos u}; \ \therefore\ \frac{\cos^2\frac{1}{2}v}{\cos^2\frac{1}{2}u} = \frac{1-e}{1 - e\cos u};$$

$$\therefore\ \cos\frac{v}{2} = \frac{\sqrt{1-e}}{\sqrt[4]{\cos\epsilon}}\cos\frac{u}{2};$$

$$\therefore\ \sin\frac{v-u}{2}=\frac{2\sin\frac{1}{2}\epsilon}{\sqrt[4]{\cos\epsilon}}\sin\frac{u}{2}\cos\frac{u}{2},$$

$$\text{or}\ \ v-u=2\sin^{-1}\left(\frac{\sin\frac{1}{2}\epsilon\sin u}{\sqrt[4]{\cos\epsilon}}\right).$$

But $u-m=e\sin u$;

$$\therefore\ v-m=e\sin u+2\sin^{-1}\left(\frac{\sin\frac{1}{2}\epsilon\sin u}{\sqrt[4]{\cos\epsilon}}\right),$$

u being known from the equation $\cos u=\dfrac{1-\sqrt[4]{1-e^2}}{e}$.

If we neglect all powers of e above the third, and call the greatest equation of the center E, we find

$$E=2e+\frac{11}{48}e^3,$$

$$\text{and}\ \ e=\frac{E}{2}-\frac{11}{48}\left(\frac{E}{2}\right)^3;$$

from the latter of which, if the greatest equation of the center be known, the eccentricity of the orbit results.

VIII. *To find when the equation of time arising from both causes in a maximum.*

Let $♈ S$ be the ecliptic, fig. 49, Plate III. $♈ D$ the equator, p the place of perigee, S that of the Sun at any time (t) from perigee, pS the true anomaly $= v$, $m =$ the corresponding mean anomaly, $p♈ = a$; $\therefore\ m-a =$ the Sun's mean longitude. Take $♈ M = m-a$, and draw SD perpendicular to the ecliptic, then M is the place of the fictitious Sun, and MD the equation of time, neglecting the motion of the equinox; let α, δ, l be the Sun's right ascension, declination, and longitude;

$$\therefore\ MD=♈ D-♈ M=a-m+a=\text{a max.};$$

$$\therefore\ d_t a-d_t m=0;\ \text{or}\ d_t m=d_t a.$$

$$\text{But}\ \ d_t a=d_t l\cos\omega\sec^2\delta=d_t v\cos\omega\sec^2\delta,\ \text{since}\ l=v-a;$$

$$\therefore\ d_v m=\cos\omega\sec^2\delta,$$

$$\text{or } \frac{(1-e^2)^{\frac{3}{2}}}{(1+e\cos v)^2} = \cos\omega \,.\, \sec^2\delta\,; \qquad (\text{Art. } 318).$$

$$\text{but } \sin\delta = \sin\omega\sin l = \sin\omega\sin(v-a),$$

$$\therefore\ (1-e^2)^{\frac{3}{2}} \,.\, \left\{1-\sin^2\omega \,.\, \sin^2(v-a)\right\} = \cos\omega\,(1+e\cos v)^2,$$

which is the equation for finding v.

IX. *To find when a planet appears stationary to the Earth, taking into account the eccentricities of the orbits, and the inclination.*

This will manifestly happen when the planet and Earth occupy such points in their respective orbits that their motions are in the same plane, (or the tangents to the orbits intersect the line of nodes in the same point), and their velocities, resolved perpendicular to the line joining them, are equal.

Let AP (fig. 54.) be a planet's orbit, $A'P'$ the Earth's, A, A', their perihelia, ST the line of nodes, PT, $P'T$, tangents intersecting the line of nodes in the same point T.

Let the argument of latitude $PST = u$, $AST = a$, $SP = r$, $SPT = \theta$; then

$$\frac{SP}{ST} = \frac{\sin(\theta+u)}{\sin\theta} = \cos u + \sin u\cot\theta,$$

$$\text{but } \cot\theta = \frac{1}{r}\,d_u r = \frac{e\sin(u-a)}{1+e\cos(u-a)},$$

$$\text{since } r = \frac{l}{1+e\cos(u-a)},$$

l and e being the $\frac{1}{2}$ latus rectum and eccentricity of the planet's orbit;

$$\therefore\ \frac{r}{ST} = \frac{\cos u + e\left\{\cos u\cos(u-a) + \sin u\sin(u-a)\right\}}{1+e\cos(u-a)}$$

$$= \frac{\cos u + e\cos a}{1+e\cos(u-a)};$$

$$\therefore\ ST = \frac{l}{\cos u + e\cos a}.$$

Similarly if u', a', represent the angular distance of the Earth, and the perihelion of its orbit from the line of nodes, and l', e', the $\frac{1}{2}$ latus rectum and eccentricity of its orbit,

$$ST = \frac{l'}{\cos u' + e' \cos a'}.$$

Hence, equating these values,

$$l(\cos u' + e' \cos a') = l'(\cos u + e \cos a). \quad (1)$$

Again, let v, v' represent the velocities of the planet and Earth, then the parts resolved perpendicular to the line PP' are $v \sin P$, $v' \sin P'$;

$$\therefore\ v \sin P = v' \sin P' \text{ or } \sqrt{l} \,.\, SY' \,.\, TP' = \sqrt{l'} \,.\, SY \,.\, TP,$$

SY, SY', being the perpendiculars let fall on TP, TP' from S; but $SY \,.\, TP = ST \,.\, SP \,.\sin u$, each being = twice the area of the triangle SPT;

$$\therefore\ \sqrt{l} \,.\, ST \,.\, SP' \sin u' = \sqrt{l'} \,.\, ST \,.\, SP \,.\sin u,$$

$$\text{or } \sqrt{l} \,.\, r' \sin u' = \sqrt{l'} \,.\, r \sin u,$$

or $\sqrt{l'} \sin u' \left\{1 + e \cos(u - a)\right\} = \sqrt{l} \sin u \left\{1 + e' \cos(u' - a')\right\}$; which, with equation (1), determines u and u' when the planet is stationary.

X. *To find when Venus appears brightest, upon the hypothesis of Venus, and the Earth describing circular orbits about the Sun.*

Let S (fig. 55.) be the Sun, P Venus, E the Earth.

Then the illuminated part of the disk varies as

$$1 - \cos pPE \propto 1 + \cos SPE,$$

and as the intensity of light from any object varies inversely as the square of the distance; therefore

$$\text{the brightness of Venus} \propto \frac{1 + \cos SPE}{PE^2}.$$

Let $PE = x$, $PS = a$, $SE = 1$; then, the brightness

$$\propto \frac{1}{x^2} \cdot \left(1 + \frac{a^2 + x^2 - 1}{2ax}\right) \propto \frac{1}{x^2} + \frac{a^2 - 1}{2ax^3} + \frac{1}{2ax};$$

therefore when the brightness is a maximum,

$$\frac{2}{x^3} + 3 \cdot \frac{a^2 - 1}{2ax^4} + \frac{1}{2ax^2} = 0;$$

$$\therefore\ x^2 + 4ax + 3(a^2 - 1) = 0;$$

$$\therefore\ x = \sqrt{a^2 + 3} - 2a,$$

the negative value of the root being inadmissible. When Venus is at the distance thus determined, which happens every eighteen months, she is sometimes visible in daylight to the naked eye; there is however a mean position, to which she returns every eight years, depending on the ratio of her periodic time to that of the Earth, in which she is brighter than in any other.

XI. *To find the retardation of the Moon's rising on successive days.*

Let HO (fig. 61.) be the horizon, EQ the equator, Z the zenith, AB the Moon's orbit, A the place of the Moon at rising on any day, and when the point A of the orbit comes again to the horizon after 24^h, suppose the Moon to have advanced in her orbit to B; then the rising will be retarded by the time the point B takes to be transferred to the horizon in b, Bb being parallel to the equator, or by the time the meridian pD takes to be brought into the position pd; pBD, pdb being circles of declination through B and b; i. e. the retardation $= Dd \div 15$.

Let AB the Moon's daily motion $= m$,

BAO the inclination of the Moon's orbit to the horizon $= n$,

and ZE the latitude of the observer $= l$.

Then considering the small triangle BAb as rectilinear,

$$Dd = Bb \sec\delta = \frac{m \sin n}{\sin BbA} \sec\delta.$$

But $\sin BbA = \cos Hpb$, and from the right-angled triangle Hpb,

$$\sin l = \sin pb \,.\, \sin Hbp = \cos \delta \,.\, \sin Hbp;$$

$$\therefore \cos Hpb = \frac{\sqrt{\cos^2 \delta - \sin^2 l}}{\cos \delta}; \qquad \therefore Dd = \frac{m \,.\, \sin n}{\sqrt{\cos^2 \delta - \sin^2 l}}.$$

Hence may be explained the phenomenon of the *Harvest Moon.* If the Moon's orbit be supposed to coincide with the ecliptic, which it does nearly, $\sin n$ is least, when the Moon rises in Aries (Art. 179.); therefore the numerator of the above expression is then least, and because $\cos^2 \delta = 1$, the denominator is then greatest; therefore on both accounts Dd is least, and if the Sun be at the same time in Libra, the Moon is then at the full; therefore the full Moon which takes place near the autumnal equinox, rises nearly at the same time for several nights, and the disappearance of the Sun is immediately succeeded by bright moonlight; and as this is near the time of harvest in north latitudes, it is called the Harvest Moon. In the same manner it may be shewn that for two or three days at the vernal equinox, the full Moon sets just at Sun-rise.

The inclination of the Moon's orbit to the horizon continually increases from the rising of its ascending node till the setting, when it is greatest; therefore, the difference of the times of setting of the full Moons, when the Sun is in Libra, is the greatest. Every time the Moon crosses the equator, her rising and setting must be similarly affected; but not being connected with the beginning or end of the day, the peculiarities at other seasons pass unnoticed.

XII. *To find the height of a lunar mountain.*

It was stated in Art. 94, that on viewing the Moon through a powerful telescope, bright spots are sometimes observed, situated slightly but distinctly beyond the inner elliptic boundary of the phase, in the unillumined part of the disk, which are supposed to be mountains; we shall now shew that if the distance of any such spot from the edge of the bright part can be measured, we may determine its height above the average spherical surface of the Moon. Let E, M, S (fig. 62.) be the centers of the Earth, Moon, and Sun, ACB a section of the

Moon perpendicular to EM, DOC a section perpendicular to SM; then the visible illumined part of the disk is contained between CBD and the projection of COD on ACB; let m be the top of a mountain so situated in the prolongation of the radius r, as to be illuminated by the ray SO which grazes the Moon's body at O, or is a tangent at O to the great circle $E'O$; then m will be seen projected on the dark part of the disk and its apparent distance from the edge of the bright part will be the projection of $mO = mO \,.\, \sin ACO$, because mO being perpendicular to the plane COD is inclined at an angle $= 90^0 - ACO$ to the plane of vision. Let the length of the projection measured by a micrometer in parts of the radius of the Moon's disk $= a$;

$$\therefore\ mO = \frac{a}{\sin ACO} = \frac{a}{\sin E},$$

E being the exterior angle of elongation; but if r be the radius of the disk, and $rm = x$, $(2r + x)\,x = (mO)^2 = a^2 \operatorname{cosec}^2 E$;

$$\therefore\ \ x = \frac{a^2}{2r} \operatorname{cosec}^2 E, \text{ neglecting } x^2;$$

where a is the observed distance of the bright spot from the illumined edge measured perpendicular to the line joining the cusps, for since SO is parallel to SM their projections will be parallel, i. e. the projection of mO will be parallel to MB.

Some of the mountains thus measured in the Moon are found to be near five miles high, which is much greater compared with the Moon's diameter, than the height of any terrestrial mountain is, compared with the Earth's.

XIII. *To find Jupiter's distance, from an observed eclipse of a satellite.*

As Jupiter's distance can hardly be determined from his parallax, which is almost insensible even when he is nearest to us, the following method of deducing it from an observed eclipse of a satellite may be employed.

Let S (fig. 63. bis) be the Sun, I Jupiter, O the satellite, and E the Earth at the time of an eclipse, which, being known, the jovicentric longitude (l) of the satellite is known.

Let the Sun's longitude, $SE\,\Upsilon = \odot$,

nI♈ or IE♈ the geocentric longitude of Jupiter $= \lambda$, $SI = r$, $SE = R$. Then $SIE = nIO = l - \lambda$, $IES = \lambda - \odot$;

$$\therefore\ r = R \cdot \frac{\sin(\lambda - \odot)}{\sin(l - \lambda)}.$$

XIV. *To measure an arc of the stereographic projection of any circle.*

Let AB (fig. 86.) be the given projection of a small circle, P that of its pole; from C the center of the primitive describe mn the projection of a circle parallel to the primitive and equal to the original of AB, and produce PA, PB through the extremities of any arc to meet the circle so described in a, b. Now if on the surface of a sphere any two circles pass through the poles of two equal circles, they will intercept equal arcs of the equal circles; therefore, because Pa, Pb are the projections of circles passing through the poles of equal circles, viz. the originals of AB, ab, they intercept equal arcs of those originals, that is, the original of the arc AB = the original of the arc ab.

If AB be the projection of a great circle, ab coincides with the primitive $a'b'$, and $a'b'$ = the original of AB. Hence, if two great circles are projected into EA, EB, (fig. 87.) and their poles into P, Q; and EP, EQ be produced to meet the primitive in p, q, the angle $AEB = pq \div r$. For E is the projection of the pole of the original of PQ, and therefore original of $PQ = pq$, or $r \cdot AEB = pq$.

XV. To construct a horizontal dial which will serve for all latitudes. Suppose the points A and B of the line AB (fig. 80. bis) moveable in two grooves CB, CA, at right angles to one another; bisect AB in M, and take

$$MH = AM \tan(45^0 - h);$$

then if the plane ACB be made horizontal, CB the substyle placed in the meridian, AB slided in such a position that $\tan B = \sin \text{lat.}$, and the style have the proper inclination given to it, CH will be the hour line corresponding to the hour angle h; for

$$\tan BCH = \frac{\sin BCH}{\sin ACH} = \frac{BH}{BC} \cdot \frac{AC}{AH} = \tan B \tan h = \sin l \tan h;$$

since $MH = AM \tan(45^0 - h)$, gives $BH = AH \tan h$.

Graduate AB by the formula $MH = AM \tan(3 - n)\,15^0$,

and in any latitude make the adjustments indicated above, then the hour will be shewn by the shadow falling on the divisions of AB.

XVI. To find the curve traced out by the extremity of the shadow of a vertical gnomon on a horizontal plane.

Let AB (fig. 84.) be the gnomon, AP the shadow, AN the direction of the meridian shadow. Draw PN perpendicular to AN, and let $AN = x$, $PN = y$, $AB = a$, l = the latitude of the place, δ = the Sun's declination. Then

$$\text{sin Sun's altitude} = \frac{AB}{BP} = \frac{a}{\sqrt{x^2 + y^2 + a^2}},$$

$$\text{cos Sun's altitude} = \frac{AP}{BP} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + a^2}},$$

$$\text{cos Sun's azimuth} = \frac{AN}{AP} = \frac{x}{\sqrt{x^2 + y^2}}.$$

But $\sin\delta = \cos l \,.\, \cos \text{alt.} \cos \text{azimuth} + \sin l \,.\, \sin \text{alt.}$ (Art. 218.);

$$\therefore\ \sin\delta = \cos l \,.\, \frac{x}{\sqrt{x^2 + y^2 + a^2}} + \frac{\sin l \,.\, a}{\sqrt{x^2 + y^2 + a^2}};$$

$$\therefore\ x^2 + y^2 + a^2 = \frac{(\cos l \,.\, x + \sin l \,.\, a)^2}{\sin^2\delta};$$

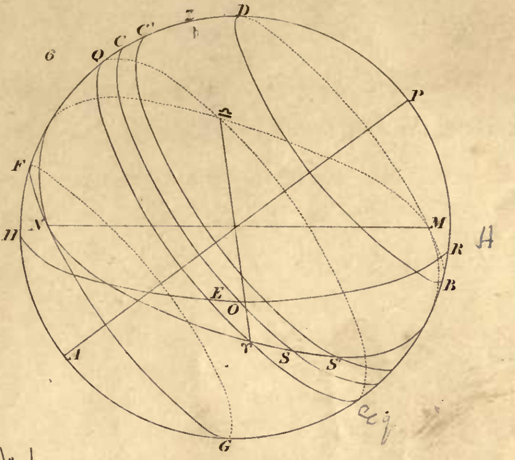
$$\therefore\ y^2 = \frac{(\cos^2 l - \sin^2\delta) \,.\, x^2 + 2a \sin l \cos l x + (\sin^2 l - \sin^2\delta) \,.\, a^2}{\sin^2\delta}.$$

If $\cos l = \sin\delta$, or $l = 90^0 - \delta$ the curve is a parabola,
$\cos l > \sin\delta$, or $l < 90^0 - \delta$.............. a hyperbola,
$\cos l < \sin\delta$, or $l > 90^0 - \delta$.............. an ellipse.

THE END.

Plate I

6



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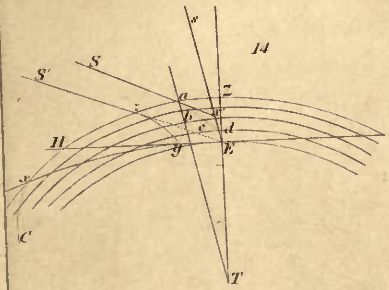

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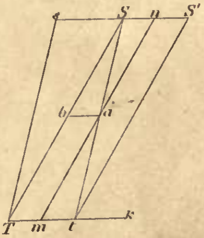

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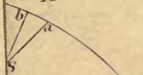

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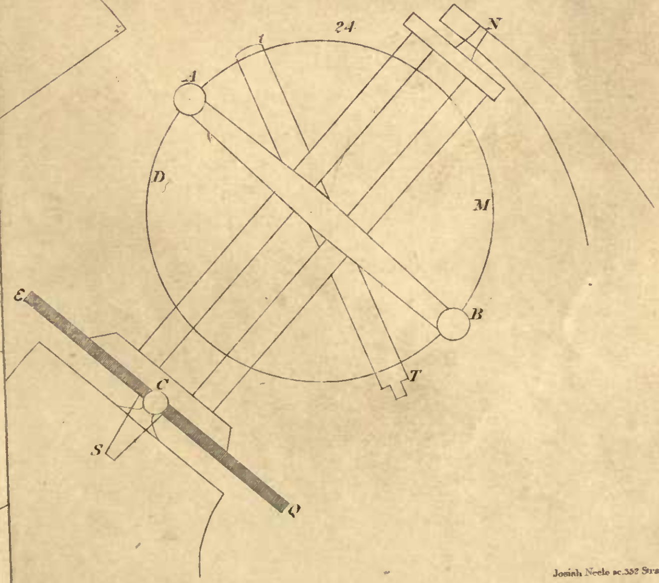

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